Causal and Strictly Causal Estimation for Jump Linear Systems: An LMI Analysis

Alyson K. Fletcher*, Sundeep Rangan†, Vivek K Goyal‡, Kannan Ramchandran*

* Dept. of Electrical Engineering and Computer Sciences
University of California, Berkeley
Berkeley, CA 94720 USA
Email: {alyson,kannanr}@eecs.berkeley.edu

†Flarion Technologies
Bedminster, NJ 07921 USA
Email: s.rangan@flarion.com

‡ Dept. of Electrical Engineering and Computer Science
Massachusetts Institute of Technology
Cambridge, MA 02139 USA
Email: vgoyal@mit.edu

***Abstract*— Jump linear systems are linear state-space systems with random time variations driven by a finite Markov chain. These models are widely used in nonlinear control, and more recently, in the study of communication over lossy channels. This paper considers a general jump linear estimation problem of estimating an unknown signal from an observed signal, where both signals are described as outputs of a jump linear system. A bound on the minimum achievable estimation error in terms of linear matrix inequalities (LMIs) is presented, along with a simple jump linear estimator that achieves this bound. While previous analysis has considered only the strictly causal estimation problem, this work presents both strictly causal and causal solutions.**

Keywords: Jump linear systems, state estimation, Kalman filtering.

I. Introduction

A *jump linear system* is a time-varying linear state-space system driven by a discrete Markov chain. Since their initial study in the 1960s, jump linear systems have found successful applications in a number of fields including fault detection in manufacturing, target tracking, guidance, finance and aerospace. See, for example, [2], [4], [5], [14], [22]. More recently, we have shown that jump linear systems can also be used to analyze the effect of losses in predictive quantization [18], [17].

This works considers a general estimation problem for a discrete-time jump linear system of the form

$$\begin{aligned} x[k+1] &= A_{\theta[k]}x[k] + B_{\theta[k]}w[k], \\ z[k] &= C_{1,\theta[k]}x[k] + D_{1,\theta[k]}w[k], \\ y[k] &= C_{2,\theta[k]}x[k] + D_{2,\theta[k]}w[k], \end{aligned} \quad (1)$$

where $w[k]$ is unit-variance, zero-mean white noise, $x[k]$ is an internal state, $z[k]$ is an output signal to be estimated, and $y[k]$ is an observed signal. The time-varying parameter $\theta[k]$ is a discrete-time Markov chain with a finite number of possible states: $\theta[k] \in \{0, \ldots, M-1\}$. All signals, $x[k]$, $w[k]$, $y[k]$ and $z[k]$ may be vector-valued, and the matrices, $A_{\theta[k]}, B_{\theta[k]}, \ldots$, are matrices of the appropriate dimensions. Thus, a jump linear system is a standard linear state-space system, where each system matrix can, at any time, take on one of M possible values. The system is called a "jump" linear system, since changes in the Markov state $\theta[k]$ result in discrete changes, or jumps, in the system dynamics.

The basic estimation problem for the system (1) is to estimate the unknown signal $z[k]$ from the observed signal $y[k]$. In this work, we assume that the estimator knows the Markov state $\theta[k]$. The estimation problem for the case when $\theta[k]$ is unknown is a complex, nonlinear estimation problem and considered elsewhere [6], [8], [15].

Under the assumption that $\theta[k]$ is known, the estimator essentially sees a linear system with known time variations. Consequently, the optimal estimate for $z[k]$ can be estimated from a standard Kalman filter [3]. Under suitable assumptions, the Kalman filter provides the MMSE estimates for the state $x[k]$ and output $z[k]$, along with corresponding estimation error variances.

However, there are two problems with the Kalman filter solution. First, the Kalman filter provides the estimation error variance sequence for a *given* realization of the Markov sequence $\theta[k]$. However, in many applications, what is relevant is not the performance of the estimator for a particular realization of the Markov sequence, but rather the *average* performance, averaged over all possible realizations of the Markov chain $\theta[k]$. Secondly, the optimal Kalman filter update involves a nonlinear Riccati matrix update. In certain real-time applications, this update may be too computationally difficult.

Following the approach of Costa in [9], we thus consider a suboptimal jump linear estimator that is similar in form to the optimal Kalman estimator; the time-varying Kalman gain matrix is replaced by a gain matrix that takes on one of M possible values, depending on the current value of the Markov state $\theta[k]$. Our main result shows that the selection of the gain matrices for this estimator can be optimized via a convex programming method based on Linear Matrix Inequalities (LMI) [7]. The optimization can be used as a bound on the minimum achievable estimation error from Kalman filtering. Alternatively, the proposed jump linear estimator is significantly simpler to implement than the optimal Kalman estimator, and the optimization can be used as a method to maximize the performance of this simplified estimator.

The LMI approach to the analysis of jump linear systems via LMIs is not new. LMIs were introduced into the study of jump linear systems by Ait Rami and El Ghaoui in [1], which provided an LMI solution for the coupled nonlinear algebraic Riccati equations earlier derived by Costa in [9]. Ait Rami and El Ghaoui's LMI considered both the state estimation considered here, as well as a related dual problem of state feedback control. Later, Costa, do Val and Geromel [10] provided a more general LMI that could be more easily extended to related problems in H_∞ control, parametric uncertainties, and game theoretic interpretations.

However, the LMI of [10] considered only the state-feedback control problem. A recent work of ours in [16] developed an analogous result for state estimation. The result was proven along similar lines as [10], beginning with the Lyapunov analysis in [11]. A similar approach has been used by De Souza and Fragoso [13] who consider the continuous-time H_∞ estimation problem. The jump linear LMI in [16] was developed in the study of estimation from intermittent data. The result is significantly more general than the LMI analyses in [19], [20], [21] that are restricted to i.i.d. losses with variations only in the output equation.

The estimation LMI in [16] and other related works, however, consider only the *strictly causal estimation* problem. That is, at each time k, it is assumed that the estimate for $z[k]$ depends only on the data $y[j]$ up to time $j = k-1$. Consequently, there is a one step lag between the observed and predicted value. The main contribution of the current paper is to provide an LMI for the *causal* estimation problem, where the estimator has access to the data, $y[j]$, up to time $j = k$. For completeness, and to facilitate comparison, both the strictly causal and causal estimation LMIs will be presented.

II. Jump Linear Estimation

A. Kalman Estimation

Before presenting the jump linear analysis, it is useful to review the Kalman filtering approach to the estimation problem. The Kalman filter equations are reviewed in a number of standard texts such as [3]. For the jump linear system (1), define the estimates

$$\begin{aligned} \hat{x}[k \mid j] &= \mathbf{E}\left(x[k] \mid y[0:j], \theta[0:k]\right), \quad (2)\\ \hat{z}[k \mid j] &= \mathbf{E}\left(z[k] \mid y[0:j], \theta[0:k]\right), \quad (3) \end{aligned}$$

which are the MMSE estimates of the state $x[k]$ and output $z[k]$ given the observed output $y[\ell]$ up to time $\ell = j$ and the Markov state $\theta[\ell]$ up to time $\ell = k$. We use the MATLAB-inspired notation $y[0:j]$ to mean the sequence, $y[0], y[1], \ldots, y[j]$. Also, let $P[k \mid j]$ be the corresponding state estimation error variance, and let $\sigma^2[k \mid j]$ be the mean-squared output estimation error,

$$\begin{aligned} P[k \mid j] &= \mathbf{E}\left(e_x[k]e_x[k]' \mid \theta[0:k]\right), \quad (4)\\ \sigma^2[k \mid j] &= \mathbf{E}\left(\|e_z[k]\|^2 \mid \theta[0:k]\right), \quad (5) \end{aligned}$$

where $e_x[k] = x[k] - \hat{x}[k \mid j]$ and $e_z[k] = z[k] - \hat{z}[k \mid j]$.

In this paper, we will be interested in two special cases:

- *strictly causal output estimation:* $\hat{z}[k \mid k-1]$, and
- *causal output estimation:* $\hat{z}[k \mid k]$.

The strictly causal estimate is also called the one-step ahead predictor, since it is estimates the output, $z[k]$, given the observed data $y[\ell]$ up to time $\ell = k-1$. The causal estimator estimates the output $z[k]$ using all the data $y[\ell]$ up to time $\ell = k$.

Now, since the Markov chain $\theta[k]$ is known to the estimator, the estimator essentially sees a linear system with known time variations. Thus, for any particular realization $\theta[k]$, we can apply the Kalman filter equations for the corresponding time-varying linear system. If the noise $w[k]$ is Gaussian, zero mean and white with unit variance, then the Kalman estimator equations for the system (1) reduce to the simple recursive form: denoting $\theta[k]$ by i,

$$\begin{aligned} \hat{x}[k+1|k] &= A_i\hat{x}[k|k-1] + L_1[k]e_y[k],\\ \hat{z}[k|k-1] &= C_{i1}\hat{x}[k|k-1],\\ \hat{z}[k|k] &= C_{i1}\hat{x}[k|k-1] + L_2[k]e_y[k],\\ e_y[k] &= y[k] - C_{i2}\hat{x}[k]. \end{aligned} \quad (6)$$

Here, $L_1[k]$ and $L_2[k]$ are time-varying Kalman gain matrices. The Kalman filter also provides a recursive equation for the estimation error variances (4) and (5).

B. Jump Linear Estimation

Unfortunately, the steady-state performance of the Kalman estimator (6) is difficult to analyze. The chief difficulty is that the Riccati equation update for $P[k|k-1]$ is a complex nonlinear recursion. As discussed in the introduction, we thus consider a simpler, suboptimal estimator. For the strictly causal estimation, we consider an estimator of the form: denoting $\theta[k]$ by i,

$$\begin{aligned} \hat{x}[k+1] &= A_i\hat{x}[k] + L_{i1}(y[k] - C_{i2}\hat{x}[k]) \\ \hat{z}[k] &= C_{i1}\hat{x}[k], \end{aligned} \tag{7}$$

defined for a set of matrices, L_{i1}, $i = 0, \ldots, M-1$. We call the estimator (7) a *strictly causal jump linear estimator*. The estimator is itself a jump linear system, driven by the Markov chain $\theta[k]$. The jump linear estimator is identical to the Kalman filter (6), except that the gain matrix $L_1[k]$ is replaced by the M fixed matrices L_{i1}. Consequently, the gain matrix for the jump linear estimator is restricted to depend on only the current value of $\theta[k]$. In contrast, the optimal gain matrix $L_1[k]$ in (6) is a result of the Riccati equation recursion and consequently depends on all the values of $\theta[j]$ from times $j = 0$ to k.

The estimator in (7) is strictly causal in that the estimate $\hat{z}[k]$ depends on the samples $y[j]$ from $j = 0$ to $k-1$. For the causal estimator, we consider a jump linear causal estimator of the form: denoting $\theta[k]$ by i,

$$\begin{aligned} \hat{x}[k+1] &= A_i\hat{x}[k] + L_{i1}(y[k] - C_{i2}\hat{x}[k]) \\ \hat{z}[k] &= C_{i1}\hat{x}[k] + L_{i2}(y[k] - C_{i2}\hat{x}[k]), \end{aligned} \tag{8}$$

The estimator is defined for a set of $2M$ gain matrices, L_{i1} and L_{i2} for $i = 0, \ldots, M-1$. Again, the estimator $\hat{z}[k]$ in (8) is identical to the estimate $\hat{z}[k|k]$ in (6), except that the time-varying gain matrices $L_1[k]$ and $L_2[k]$ are replaced by the $2M$ fixed gain matrices L_{i1} and L_{i2}, $i = 0, \ldots, M-1$.

III. LMI Optimization

A. Asymptotic Estimation Error

In order to apply the jump linear analysis, we make the following assumption on the Markov chain $\theta[k]$.

Assumption 1: The random sequence $\theta[k]$ is a stationary Markov chain, with transition probabilities,

$$p_{ij} = \mathbf{Pr}\left(\theta[k+1] = j \mid \theta[k] = i\right).$$

We assume that the Markov chain is aperiodic and irreducible so that there is a unique stationary distribution,

$$q_i = \mathbf{Pr}\left(\theta[k] = i\right),$$

satisfying the equations

$$q_j = \sum_{i=0}^{M-1} p_{ij} q_i, \quad j = 0, \ldots, M-1.$$

Both the strictly causal estimator (7) and causal estimator (8) can be analyzed relatively easily using LMIs. Let

$$\begin{aligned} \mathbf{L}_1 &= (L_{0,1}, \ldots, L_{M-1,1}), \\ \mathbf{L}_2 &= (L_{0,2}, \ldots, L_{M-1,2}), \end{aligned} \tag{9}$$

denote the set of gain matrices for the estimators. For the strictly causal estimator (7), define the asymptotic error variance

$$\sigma^2(\mathbf{L}_1) = \lim_{k\to\infty} \mathbf{E}\|z[k] - \hat{z}[k]\|^2, \tag{10}$$

where the dependence on $\mathbf{L}_1$ is through the estimate $\hat{z}[k]$. The LMI analysis will attempt to find the set of matrices $\mathbf{L}_1$ that minimizes this asymptotic error:

$$\min_{\mathbf{L}_1} \sigma^2(\mathbf{L}_1). \tag{11}$$

That is, we regard the matrices $\mathbf{L}_1$ as a set of design parameters for the estimator (7), and attempt to optimize the estimator performance. The use of the limit in (10) is to eliminate the effect of initial conditions.

Similarly, for the causal estimator, (8), we can define the asymptotic error variance

$$\sigma^2(\mathbf{L}_1, \mathbf{L}_2) = \lim_{k\to\infty} \mathbf{E}\|z[k] - \hat{z}[k]\|^2, \tag{12}$$

where, again, the dependence on $\mathbf{L}_1$ and $\mathbf{L}_2$ is through the estimate $\hat{z}[k]$ in (8). As in the strictly causal case, we seek to find gain matrices, $\mathbf{L}_1$ and $\mathbf{L}_2$ to minimize $\sigma^2(\mathbf{L}_1, \mathbf{L}_2)$:

$$\min_{\mathbf{L}_1, \mathbf{L}_2} \sigma^2(\mathbf{L}_1, \mathbf{L}_2). \tag{13}$$

We can regard the optimization (11) in one of two ways. On the one hand, we can regard the optimization as a design method to maximize the performance of the estimator (7). The resulting estimator is useful in its own right. Although the estimator (7) is not optimal, it is significantly simpler to implement than the optimal Kalman filter (6). In each iteration of the Kalman filter, one must perform the Riccati recursion to update the state variance and compute $L_{i1}[k]$. In the simplified estimator (7), the gain matrices L_{i1} can be pre-computed and do not require any real-time computations. If gain matrices L_{i1} can be found with suitable performance, one can realize significant computational savings. Similarly, minimizing $\sigma^2(\mathbf{L}_1, \mathbf{L}_2)$ optimizes a simplified causal estimator (8).

Alternatively, we can use the minimization (11) as an analytic tool. Specifically, since the Kalman filter is optimal, the performance of any estimator of the form (7) provides an *upper bound* on the Kalman filter performance. Performing the minimization (11) minimizes that upper bound.

In either interpretation of the optimization, we will restrict our attention to mean-square (MS) stabilizing gain matrices $\mathbf{L}$ as defined in Definition 2 in the Appendix. The restriction guarantees that the state estimate error is bounded—a mild assumption.

B. Strictly Causal Optimization

With the above definitions, we can now present an LMI solution to the jump linear estimation problem. We begin with the strictly causal optimization problem (11).

Theorem 1: Consider the jump linear system in (1) driven by a Markov chain $\theta[k]$ satisfying Assumption 1. Suppose $w[k]$ is zero-mean, white noise with unit variance independent of $\theta[k]$. Given a set of gain matrices, L_1 in (9), let $\hat{z}[k]$ be the estimator output in (7), and let $\sigma^2(L_1)$ be asymptotic mean-squared output estimation error (10).

(a) Suppose that C_{i1} is injective for all i, and suppose that there exist matrices W_i, $i = 0, \ldots, M-1$, partitioned as

$$W_i = \begin{bmatrix} W_{i1} & W_{i2} \\ W_{i2}' & W_{i3} \end{bmatrix}, \tag{14}$$

satisfying

$$\begin{array}{rcl} W_{i1} & \geq & [A_i' \ C_{i2}']\overline{W}_i[A_i' \ C_{i2}']' + C_{i1}'C_{i1}, \\ \overline{W}_i & \geq & 0 \end{array} \tag{15}$$

where $\overline{W}_i$ is defined by

$$\overline{W}_i = \begin{bmatrix} \overline{W}_{i1} & W_{i2} \\ W_{i2}' & W_{i3} \end{bmatrix}, \tag{16}$$

and

$$\overline{W}_{i1} = \sum_{j=0}^{M-1} p_{ij} W_{j1}. \tag{17}$$

Then, $\overline{W}_{i1} > 0$ for all i. Also, if we define

$$L_{i1} = -\overline{W}_{i1}^{-1} W_{i2}, \tag{18}$$

the set of matrices L_1 is MS stabilizing and the asymptotic mean-squared error is bounded by

$$\sigma^2(L) \leq \sum_{i=0}^{M-1} q_i \mathbf{Tr}\left(E_i'\overline{W}_i E_i + D_{i1}' D_{i1}\right), \tag{19}$$

where $E_i = [B_i' \ D_{i2}']'$.

(b) Conversely, for any set of MS stabilizing gain matrices L_1, there must exist W_i satisfying (14) and (15) with

$$\sum_{i=0}^{M-1} q_i \mathbf{Tr}\left(E_i'\overline{W}_i E_i + D_{i1}' D_{i1}\right) \leq \sigma^2(L_1). \tag{20}$$

For space considerations, we will omit the proof of the theorem. The result is proven along lines similar to Theorem 2, whose proof is presented.

Combining parts (a) and (b) of Theorem 1, we see that the minimum asymptotic estimation error is given by

$$\min_{L_1} \sigma^2(L_1) = \min_{W_i} \sum_{i=0}^{M-1} q_i \mathbf{Tr}\left(E_i' W_i E_i + D_{i1}' D_{i1}\right), \tag{21}$$

where the first minimization is over MS stabilizing gain matrices L_1, and the second minimization is over matrices W_i satisfying (14) and (15). For a fixed set of transition probabilities, p_{ij}, the objective function (21) and constraint (15) are linear in the variables W_i. Consequently, the optimization can be solved as an LMI, thus providing a simple way to optimize the jump linear estimator.

C. Causal Estimation Optimization

We next consider the causal estimator optimization (13).

Theorem 2: Consider the jump linear system in (1) driven by a Markov chain $\theta[k]$ satisfying Assumption 1. Suppose $w[k]$ is zero-mean, white noise with unit variance independent of $\theta[k]$. Given sets of gain matrices (L_1, L_2) as in (9), let $\hat{z}[k]$ be the estimator output in (8), and let $\sigma^2(L_1, L_2)$ be asymptotic mean-squared output estimation error (12).

(a) Suppose that $[C_{i1}' \ C_{i2}']$ is onto for all i, and suppose that there exist matrices W_i and V_i, $i = 0, \ldots, M-1$, partitioned as

$$W_i = \begin{bmatrix} W_{i1} & W_{i2} \\ W_{i2}' & W_{i3} \end{bmatrix}, \quad V_i = \begin{bmatrix} I & V_{i2} \\ V_{i2}' & V_{i3} \end{bmatrix}, \tag{22}$$

satisfying

$$\begin{array}{rcl} W_{i1} & \geq & [A_i' \ C_{i2}']\overline{W}_i[A_i' \ C_{i2}']' \\ & & + [C_{i1}' C_{i2}']V_i[\ C_{i1}' \ C_{i2}']', \\ \overline{W}_i & \geq & 0, \\ V_i & \geq & 0, \end{array} \tag{23}$$

where $\overline{W}_i$ is defined in (16). Then, $\overline{W}_{i1} > 0$ for all i. Also, if we define

$$L_{i1} = -\overline{W}_{i1}^{-1} W_{i2}, \quad L_{i2} = -V_{i2}, \tag{24}$$

the set of matrices (L_1, L_2) is MS stabilizing and the asymptotic mean-squared error is bounded by

$$\sigma^2(L_1, L_2) \leq \sum_{i=0}^{M-1} q_i \mathbf{Tr}(E_i'\overline{W}_i E_i + F_i' V_i F_i), \tag{25}$$

where $E_i = [B_i' \ D_{i2}']'$ and $F_i = [D_{i1}' \ D_{i2}']'$.

(b) Conversely, for any set of MS stabilizing gain matrices (L_1, L_2), there must exist matrices W_i and V_i satisfying (23) and

$$\sum_{i=0}^{M-1} q_i \mathbf{Tr}(E_i'\overline{W}_i E_i + F_i' V_i F_i) \leq \sigma^2(L_1, L_2). \tag{26}$$

The LMI optimization in Theorem 2 is similar to Theorem 1, except that the optimization has a set of variables, V_i, in addition to the variables W_i.

Proof of Theorem 2: For any estimator of the form (7), we define the error signals,

$$e_x[k] = x[k] - \hat{x}[k], \quad e_z[k] = z[k] - \hat{z}[k].$$

Then, subtracting (1) and (8), we obtain the closed-loop system

$$\begin{aligned} e_x[k+1] &= A_{Li} e_x[k] + B_{Li} w[k], \\ e_z[k] &= C_{Li} e_x[k] + D_{Li} w[k], \end{aligned}$$

with the closed-loop matrices

$$\begin{aligned} A_{Li} &= A_i - L_{i1} C_{i2}, \\ B_{Li} &= B_i - L_{i1} D_{i2}, \\ C_{Li} &= C_{i1} - L_{i2} C_{i2}, \\ D_{Li} &= D_{i1} - L_{i2} D_{i2}. \end{aligned}$$

Also, the asymptotic mean-squared error is

$$\sigma^2(L_1, L_2) = \lim_{k \to \infty} \mathbf{E} \| z[k] - \hat{z}[k] \|^2 = \lim_{k \to \infty} \mathbf{E} \| e_z[k] \|^2.$$

Now, suppose there exists a matrix W_i and V_i as in part (a) of the theorem. We will first prove that $\overline{W}_{i1} > 0$. Observe that

$$\begin{aligned} & [C_{i1}' \;\; C_{i2}']' V_i [C_{i1}' \;\; C_{i2}']' \\ &= C_{i1}' C_{i1} + C_{i1}' V_{i2} C_{i2} + C_{i2}' V_{i2}' C_{i1} + C_{i2}' V_{i3} C_{i2} \\ &= (C_{i1} + V_{i2} C_{i2})' (C_{i1} + V_{i2} C_{i2}) \\ &\quad + C_{i2}' (V_{i3} - V_{i2}' V_{i2}) C_{i2} \\ &\geq (C_{i1} + V_{i2} C_{i2})' (C_{i1} + V_{i2} C_{i2}), \end{aligned}$$

where in the last step we have use the Schur complement on the matrix $V \geq 0$ to show that $V_{i3} \geq V_{i2}' V_{i2}$. Also, note that

$$C_{i1} + V_{i2} C_{i2} = [I \;\; V_{i2}] [C_{i1}' \;\; C_{i2}']'.$$

Now, by the assumption of the theorem, $[C_{i1}' \;\; C_{i2}']'$ is injective, as is the matrix $[I \;\; V_{i2}]$. Since the product of injective matrices is injective, $C_{i1} + V_{i2} C_{i2}$ is injective, and therefore, $(C_{i1} + V_{i2} C_{i2})' (C_{i1} + V_{i2} C_{i2}) > 0$. It follows from (27) that

$$[C_{i1}' \;\; C_{i2}']' V_i [C_{i1}' \;\; C_{i2}'] > 0.$$

Combining this with the fact that $\overline{W}_i \geq 0$, (23) shows that $W_{i1} > 0$. Consequently, W_{i1} is invertible and we can define L_{i1} and L_{i2} as in (24).

Then, if we let $Q_i = W_{i1}$ and define L_{i1} and L_{i2} as in (24), it can be verified that

$$Q_i \geq A_{Li}' \overline{Q}_i A_{Li} + C_{Li}' C_{Li},$$

so Q_i satisfies the Lyapunov equation for the closed-loop system. Also, $C_{Li} = C_{i1} - L_{i2} C_{i2} = C_{i1} + V_{i2} C_{i2}$, which is injective. Consequently, (A_{Li}, C_{Li}) is detectable. Therefore, by Proposition 1(b) (in the Appendix), $A_{Li} = A_i - L_{i1} C_{i2}$ must be MS stable, and L_{i1} is MS stabilizing. Moreover, using similar calculations, one can show that the asymptotic mean-squared error is bounded by

$$\begin{aligned} \sigma^2(L_1, L_2) &\leq \mathbf{Tr} \left(B_{Li}' \overline{Q}_i B_{Li} + D_{Li}' D_{Li} \right) \\ &\leq \mathbf{Tr} \left(E_i' \overline{W}_i E_i + F_i' V_i F_i \right), \end{aligned}$$

which proves part (a).

Conversely, consider any stabilizing matrices L_{i1} and L_{i2}. Then, the closed-loop matrix $A_i - L_{i1} C_{i2}$ must be MS stable. Consequently, by Proposition 1, there exists a $Q_i \geq 0$ satisfying the Lyapunov equation for the closed-loop system

$$Q_i = A_{Li}' \overline{Q}_i A_{Li} + C_{Li}' C_{Li},$$

with

$$\sigma^2(L_{i1}, L_{i2}) = \mathbf{Tr} \left(B_{Li}' \overline{Q}_i B_{Li} + D_{Li}' D_{Li} \right).$$

If we define W_i and V_i as in (22) with

$$\begin{aligned} W_{i1} &= Q_i, \quad W_{i2} = -Q_i L_{i1}, \quad W_{i3} = L_{i1}' Q_i L_{i1}, \\ V_{i2} &= -L_{i2}, \quad V_{i3} = L_{i2}' L_{i2}, \end{aligned}$$

then it can be verified that $\overline{W}_i \geq 0$ and $V_i \geq 0$. Also, using similar calculations as before, one can show that W_i and V_i satisfy (23), which proves part (b). □

APPENDIX: JUMP LINEAR STABILITY AND LYAPUNOV ANALYSIS

We review some standard properties and definitions for jump linear systems from [11]. The material is also covered in [9] and the book [12]. Consider a jump linear system of the form

$$\begin{aligned} x[k+1] &= A_{\theta[k]} x[k] + B_{\theta[k]} w[k], \\ z[k] &= C_{\theta[k]} x[k] + D_{\theta[k]} w[k], \end{aligned} \tag{27}$$

where $\theta[k]$ is some Markov chain satisfying Assumption 1. The following definition is a stochastic version of the standard notion of internal stability for LTI systems.

Definition 1: Consider the jump linear system in (27) driven by a Markov chain $\theta[k]$. We will say the system is *mean square (MS) stable* if: for any initial condition, $(x[0], \theta[0])$,

$$\lim_{k \to \infty} \mathbf{E} \| x[k] \|^2 = 0$$

whenever $w[k] = 0$. The expectation here is with respect to the realizations of the Markov chain $\theta[k]$. We will say that a set of matrices A_i, $i = 0, \ldots, M-1$, are *MS stable with respect to the Markov chain* $\theta[k]$ if the corresponding jump linear system (27) is MS stable.

It is well-known that stability of an LTI state-space system is equivalent to the existence of a solution to a certain Lyapunov equation. The Lyapunov equation can also be used to compute the steady-state output variance. A well-known result due to Costa and Fragosa in [11] shows that a similar Lyapunov analysis can be performed for jump linear systems. To state the result, let

$$\hat{z}[k] = \mathbf{E}z[k],$$

where the expectation is over both the noise $w[k]$ *and* the discrete state sequence $\theta[k]$. Let $\sigma^2[k]$ denote the output variance,

$$\sigma^2[k] = \mathbf{E}\|z[k] - \hat{z}[k]\|^2. \quad (28)$$

The following is proven in [11].

Proposition 1: Consider the jump linear system in (27) driven by a Markov chain $\theta[k]$ satisfying Assumption 1. Suppose that the input $w[k]$ is a zero-mean, white random process with unit variance. Then:

(a) If the system is MS stable, there exist matrices $Q_j \geq 0$, $j = 0, \ldots, M-1$ satisfying

$$Q_i = A_i'\overline{Q}_i A_i + C_i'C_i, \quad i = 0, \ldots, M-1, \quad (29)$$

where

$$\overline{Q}_i = \sum_{j=0}^{M-1} p_{ij} Q_j. \quad (30)$$

The asymptotic output variance is given by

$$\lim_{k\to\infty} \sigma^2[k] = \sum_{j=0}^{M-1} q_i \mathbf{Tr}\left(B_i'\overline{Q}_i B_i + D_i'D_i\right).$$

(b) Conversely, suppose $C_i'C_i > 0$ for all i, and there exist matrices $Q_j \geq 0$ satisfying

$$Q_i \geq A_i'\overline{Q}_i A_i + C_i'C_i, \quad i = 0, \ldots, M-1, \quad (31)$$

where $\overline{Q}_i$ is defined as in (30). Then the system is MS stable and

$$\lim_{k\to\infty} \sigma^2[k] \leq \sum_{j=0}^{M-1} q_i \mathbf{Tr}\left(B_i'\overline{Q}_i B_i + D_i'D_i\right).$$

The equations (29) are a natural generalization of the Lyapunov equations for LTI systems, with the main difference being that there are M such equations, one equation for each discrete state. The equations are called *coupled* since each equation depends on all possible values of Q_j through the term $\overline{Q}_j$.

We conclude with a jump linear version of detectability.

Definition 2: Consider the jump linear system in (27) driven by a Markov chain $\theta[k]$. The matrices (A_i, C_{i2}) will be called *mean square (MS) detectable* with respect to $\theta[k]$ if there exist matrices L_{i1} such that the set of matrices $A_i - L_{i1}iC_{i2}$ is MS stable. Any set of gain matrices L_{i1} resulting in $A_i - L_{i1}iC_{i2}$ being MS stable will be called *MS stabilizing*.

Again, this definition of detectability is a natural extension of the notion for LTI systems, and will play a similar role in estimation.

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