

Ranked Sparse Signal Support Detection

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Abstract—This paper considers the problem of detecting the support (sparsity pattern) of a sparse vector from random noisy measurements. Conditional power of a component of the sparse vector is defined as the energy conditioned on the component being nonzero. Analysis of a simplified version of orthogonal matching pursuit (OMP) called sequential OMP (SequOMP) demonstrates the importance of knowledge of the rankings of conditional powers. When the simple SequOMP algorithm is applied to components in nonincreasing order of conditional power, the detrimental effect of dynamic range on thresholding performance is eliminated. Furthermore, under the most favorable conditional powers, the performance of SequOMP approaches maximum likelihood performance at high signal-to-noise ratio.

Index Terms—Compressed sensing, convex optimization, lasso, maximum likelihood estimation, orthogonal matching pursuit, random matrices, sparse Bayesian learning, sparsity, thresholding.

I. INTRODUCTION

SETS of signals that are sparse or approximately sparse with respect to some basis are ubiquitous because signal modeling often has the implicit goal of finding such bases. Using a sparsifying basis, a simple abstraction that applies in many settings is for

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{d} \quad (1)$$

to be observed, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is known, $\mathbf{x} \in \mathbb{R}^n$ is the unknown sparse signal of interest, and $\mathbf{d} \in \mathbb{R}^m$ is random noise. When $m < n$, constraints or prior information about \mathbf{x} are essential to both estimation (finding vector $\hat{\mathbf{x}}(\mathbf{y})$ such that $\|\mathbf{x} - \hat{\mathbf{x}}\|$ is small) and detection (finding index set $\hat{I}(\mathbf{y})$ equal to the support of \mathbf{x}). The focus of this paper is on the use of magnitude rank information on \mathbf{x} —in addition to sparsity—in the support detection problem. We show that certain scaling laws relating the problem dimensions and the noise level are changed dramatically by exploiting the rank information in a simple sequential detection algorithm.

The simplicity of the observation model (1) belies the variety of questions that can be posed and the difficulty of precise analysis. In general, the performance of any algorithm is a complicated function of \mathbf{A} , \mathbf{x} , and the distribution of \mathbf{d} . To enable

results that show the qualitative behavior in terms of problem dimensions and a few other parameters, we assume the entries of \mathbf{A} are i.i.d. normal, and we describe \mathbf{x} by its energy and its smallest-magnitude nonzero entry.

We consider a partially-random signal model

$$x_j = b_j s_j, \quad j = 1, 2, \dots, n, \quad (2)$$

where components of vector \mathbf{b} are i.i.d. Bernoulli random variables with $\Pr(b_j = 1) = 1 - \Pr(b_j = 0) = \lambda > 0$ and \mathbf{s} is a nonrandom parameter vector with all nonzero entries. The value s_j^2 represents the *conditional power* of the component x_j in the event that $b_j = 1$. We consider the problem where the estimator knows neither b_j nor s_j , but may know the *order* or *rank* of the conditional powers. In this case, the estimator can, for example, sort the components of \mathbf{s} in an order such that

$$|s_1| \geq |s_2| \geq \dots \geq |s_n| > 0. \quad (3)$$

A stylized application in which the conditional ranks (and furthermore approximate conditional powers) can be known is random access communication as described in [1]. Also, partial orders of conditional powers can be known in some applications because of the magnitude variation of wavelet coefficients across scale [2]. Along with being motivated by these applications, we aim to provide a new theoretical grounding for a known empirical phenomenon: orthogonal matching pursuit (OMP) and sparse Bayesian learning (see references below) exhibit improvements in detection performance when the nonzero entries of the signal have higher dynamic range.

A. Main Contribution

Rank information is extremely valuable in support detection. Abstracting from the applications above, we show that when conditional rank information is available, a very simple detector, termed *sequential orthogonal matching pursuit* (SequOMP), can be effective. The SequOMP algorithm is a one-pass version of the well-known OMP algorithm. Similar to several works in sparsity pattern recovery [3]–[5], we analyze the performance of SequOMP by estimating a scaling on the minimum number of measurements m to asymptotically reliably detect the sparsity pattern (support) of \mathbf{x} in the limit of large random matrices \mathbf{A} . Although the SequOMP algorithm is extremely simple, we show:

- When the power orders are known and the signal-to-noise ratio (SNR) is high, the SequOMP algorithm exhibits a scaling in the minimum number of measurements for sparsity pattern recovery that is within a constant factor of the more sophisticated lasso and OMP algorithms. In particular, SequOMP exhibits a resistance to large dynamic ranges, which is one of the main motivations for using lasso and OMP.
- When the power profile can be optimized, SequOMP can achieve measurement scaling for sparsity pattern recovery

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TABLE I
SUMMARY OF RESULTS ON MEASUREMENT SCALINGS FOR ASYMPTOTIC RELIABLE DETECTION FOR VARIOUS DETECTION ALGORITHMS. ONLY LEADING TERMS ARE SHOWN. SEE BODY FOR DEFINITIONS AND TECHNICAL LIMITATIONS

	finite SNR · MAR	SNR · MAR → ∞
Necessary for ML	$m > \frac{2}{\text{MAR} \cdot \text{SNR}} k \log(n - k)$ Fletcher et al. [5, Thm. 1]	$m > k$ (elementary)
Sufficient for ML	$m > \frac{C}{\text{MAR} \cdot \text{SNR}} k \log(n - k)$ Wainwright [3]	$m > k$ (elementary)
Sufficient for SequOMP with best power profile	$m > \frac{8}{\log(1 + \text{SNR})} k \log(n - k)$ From Theorem 1 (Section IV-D)	$m > 9k$ From Theorem 1 (Section IV-E)
Sufficient for SequOMP with known conditional ranks	$m > \frac{8(1 + \text{SNR} \cdot \text{MAR})}{\text{SNR} \cdot \text{MAR}} k \log(n - k)$ From Theorem 1 (Section IV-C)	$m > 8k \log(n - k)$ From Theorem 1 (Section IV-C)
Necessary and sufficient for lasso	complicated; see [4]	$m > 2k \log(n - k)$ Wainwright [4]
Sufficient for OMP	unknown	$m > 2k \log(n - k)$ Fletcher and Rangan [29]
Sufficient for thresholding (12)	$m > \frac{8(1 + \text{SNR})}{\text{MAR} \cdot \text{SNR}} k \log(n - k)$ Fletcher et al. [5, Thm. 2]	$m > \frac{8}{\text{MAR}} k \log(n - k)$

that is within a constant factor of maximum likelihood (ML) detection. This scaling is better than the best known sufficient conditions for lasso and OMP.

The results are not meant to suggest that SequOMP is a good algorithm; other algorithms such as OMP can perform dramatically better. The point is to concretely and provably demonstrate the value of conditional rank information.

B. Related Work

Under an i.i.d. Gaussian assumption on \mathbf{d} , maximum likelihood estimation of \mathbf{x} under a sparsity constraint is equivalent to finding sparse $\hat{\mathbf{x}}$ such that $\|\mathbf{y} - \mathbf{A}\hat{\mathbf{x}}\|_2$ is minimized. This is called optimal sparse approximation of \mathbf{y} using dictionary \mathbf{A} , and it is NP-hard [6]. Several greedy heuristics (matching pursuit [7] and its variants with orthogonalization [8]–[10] and iterative refinement [11], [12]) and convex relaxations (basis pursuit [13], lasso [14], Dantzig selector [15], and others) have been developed for sparse approximation, and under certain conditions on \mathbf{A} and \mathbf{y} they give optimal or near-optimal performance [16]–[18]. Results showing that near-optimal estimation of \mathbf{x} is obtained with convex relaxations, pointwise over compressible \mathbf{x} and with high probability over some random ensemble for \mathbf{A} , form the heart of the compressed sensing literature [19]–[21]. Under a probabilistic model for \mathbf{x} and certain additional assumptions, exact asymptotic performances of several estimators are known [22].

Our interest is in recovery or detection of the support (or *sparsity pattern*) of \mathbf{x} rather than the estimation of \mathbf{x} . In the noiseless case of $\mathbf{d} = 0$, optimal estimation of \mathbf{x} can yield $\hat{\mathbf{x}} = \mathbf{x}$ under certain conditions on \mathbf{A} ; estimation and detection then coincide, and some papers cited above and notably [23] contain relevant results. In the general noisy case, direct analysis of the detection problem has yielded much sharper results.

A standard formulation is to treat \mathbf{s} as a nonrandom parameter vector and \mathbf{b} as either nonrandom with weight k or random with

a uniform distribution over the weight- k vectors. The minimum probability of detection error is then attained with ML detection. Sufficient conditions for the success of ML detection are due to Wainwright [3]; necessary conditions based on channel capacity were given by several authors [24]–[27], and conditions more stringent in many regimes and a comparison of results appear in [5]. Necessary and sufficient conditions for lasso were determined by Wainwright [4]. Sufficient conditions for orthogonal matching pursuit (OMP) were given by Tropp and Gilbert [28] and improved by Fletcher and Rangan [29]. Even simpler than OMP is a thresholding algorithm analyzed in a noiseless setting in [30] and with noise in [5]. These results are summarized in Table I, using terminology defined formally in Section II. While thresholded backprojection is unsophisticated from a signal processing point of view, it is simple and commonly used in a variety of fields. Improvements relative to this are needed to justify the use of methods with higher complexity.

Some of our results depend on knowledge of the ordering of conditional powers of entries of \mathbf{x} . Several earlier works have introduced other models of partial information about signal support or varying likelihoods of indexes appearing in the support [31]–[33]. Statistical dependencies between components of \mathbf{x} can be exploited very efficiently using a recent extension [34] of the generalized approximate message passing framework [35].

C. Paper Organization

The remainder of the paper is organized as follows. The setting is formalized in Section II. In particular, we define all the key problem parameters. Common algorithms and previous results on their performances are then presented in Section III. We will see that there is a potentially-large performance gap between the simplest thresholding algorithm and the optimal ML detection, depending on the signal-to-noise ratio (SNR) and the dynamic range of \mathbf{x} . Section IV presents a new detection algorithm, sequential orthogonal matching pursuit (SequOMP),

that exploits knowledge of conditional ranks. Numerical experiments are reported in Section V. Conclusions are given in Section VI, and proofs are relegated to the Appendix.

II. PROBLEM FORMULATION

In the observation model $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{d}$, let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{d} \in \mathbb{R}^m$ have i.i.d. $\mathcal{N}(0, 1/m)$ entries. This is a normalization under which the ratio of conditional total signal energy to total noise energy

$$\text{SNR}(\mathbf{x}) = \frac{\mathbf{E}[\|\mathbf{A}\mathbf{x}\|^2 \mid \mathbf{x}]}{\mathbf{E}[\|\mathbf{d}\|^2]} \quad (4)$$

simplifies to

$$\text{SNR}(\mathbf{x}) = \|\mathbf{x}\|^2. \quad (5)$$

This is a random variable because \mathbf{x} is a random vector.

Let $I_{\text{true}} = \{j \in \{1, 2, \dots, n\} : x_j \neq 0\}$ denote the *support* of \mathbf{x} . Using signal model (2), $I_{\text{true}} = \{j \in \{1, 2, \dots, n\} : b_j = 1\}$. The *sparsity level* of \mathbf{x} is $k = |I_{\text{true}}|$.

An estimator produces an estimate $\hat{I} = \hat{I}(\mathbf{y})$ of I_{true} based on the observed noisy vector \mathbf{y} . Given an estimator, its probability of error¹ $p_{\text{err}} = \Pr(\hat{I} \neq I_{\text{true}})$ is taken with respect to randomness in \mathbf{A} , noise vector \mathbf{d} , and signal \mathbf{x} . Our interest is in relating the scaling of problem parameters with the success of various algorithms. For this, we define the following criterion.

Definition 1: Suppose that we are given deterministic sequences $m = m(n)$, $\lambda = \lambda(n)$, and $\mathbf{s} = \mathbf{s}(n) \in \mathbb{R}^n$ that vary with n . For a given detection algorithm $\hat{I} = \hat{I}(\mathbf{y})$, the probability of error p_{err} is some function of n . We say that the detection algorithm achieves *asymptotic reliable detection* when $\lim_{n \rightarrow \infty} p_{\text{err}}(n) = 0$.

We will see that two key factors influence the ability to detect I_{true} . The first is the total SNR defined above. The second is what we call the *minimum-to-average ratio*

$$\text{MAR}(\mathbf{x}) = \frac{\min_{j \in I_{\text{true}}} |x_j|^2}{\|\mathbf{x}\|^2/k}. \quad (6)$$

Like $\text{SNR}(\mathbf{x})$, this is a random variable. Since I_{true} has k elements, $\|\mathbf{x}\|^2/k$ is the average of $\{|x_j|^2 : j \in I_{\text{true}}\}$. Therefore, $\text{MAR}(\mathbf{x}) \in (0, 1]$ with the upper limit occurring when all the nonzero entries of \mathbf{x} have the same magnitude.

Finally, we define the *minimum component SNR* to be

$$\text{SNR}_{\min}(\mathbf{x}) = \frac{\min_{j \in I_{\text{true}}} \mathbf{E}[\|\mathbf{a}_j x_j\|^2 \mid \mathbf{x}]}{\mathbf{E}[\|\mathbf{d}\|^2]} = \min_{j \in I_{\text{true}}} |x_j|^2, \quad (7)$$

where \mathbf{a}_j is the j th column of \mathbf{A} and the second equality follows from the normalization of chosen for \mathbf{A} and \mathbf{d} . The random variable $\text{SNR}_{\min}(\mathbf{x})$ has a natural interpretation: The numerator is the signal power due to the smallest nonzero component in \mathbf{x} , while the denominator is the total noise power. The ratio $\text{SNR}_{\min}(\mathbf{x})$ thus represents the contribution to the SNR from the smallest nonzero component of \mathbf{x} . Observe that (5) and (6) show

$$\text{SNR}_{\min}(\mathbf{x}) = \min_{j \in I_{\text{true}}} |x_j|^2 = \frac{1}{k} \text{SNR}(\mathbf{x}) \cdot \text{MAR}(\mathbf{x}). \quad (8)$$

¹An alternative to this definition of p_{err} could be to allow a nonzero fraction of detection errors [26], [27].

We will be interested in estimators that exploit minimal prior knowledge on \mathbf{x} : either only knowledge of sparsity level (through k or λ) or also knowledge of the conditional ranks (through the imposition of (3)). In particular, full knowledge of \mathbf{s} would change the problem considerably because the finite number of possibilities for \mathbf{x} could be exploited.

III. COMMON DETECTION METHODS

In this section, we review several asymptotic analyses for detection of sparse signal support. These previous results hold *pointwise* over sequences of problems of increasing dimension n , i.e., treating \mathbf{x} as an unknown deterministic quantity. That makes these results *stronger* than results that are limited to the model (2) where the b_j s are i.i.d. Bernoulli variables. To reflect the pointwise validity of these results, they are stated in terms of deterministic sequences \mathbf{x} , m , k , SNR, MAR, and SNR_{\min} that depend on dimension n and are arbitrary aside from satisfying $m \rightarrow \infty$ and the definitions of the previous section. To simplify the notation, we drop the dependence of \mathbf{x} , m and k on n , and SNR, MAR and SNR_{\min} on $\mathbf{x}(n)$. When the results are tabulated for comparison with each other and with the results of Section IV, we replace k with λn ; this specializes the results to the model (2).

A. Optimal Detection With No Noise

To understand the limits of detection, it is useful to first consider the minimum number of measurements when there is no noise. Suppose that k is known to the detector. With no noise, the observed vector is $\mathbf{y} = \mathbf{A}\mathbf{x}$, which will belong to one of $J = \binom{n}{k}$ subspaces spanned by k columns of \mathbf{A} . If $m > k$, then these subspaces will be distinct with probability 1. Thus, an exhaustive search through the subspaces will reveal which subspace \mathbf{y} belongs to and thus determine the support I_{true} . This shows that with no noise and no computational limits, the scaling in measurements of

$$m > k \quad (9)$$

is sufficient for asymptotic reliable detection.

Conversely, if no prior information is known at the detector other than \mathbf{x} being k -sparse, then the condition (9) is also necessary. If $m \leq k$, then for almost all \mathbf{A} , any k columns of \mathbf{A} span \mathbb{R}^m . Consequently, any observed vector $\mathbf{y} = \mathbf{A}\mathbf{x}$ is consistent with any support of weight k . Thus, the support cannot be determined without further prior information on the signal \mathbf{x} .

Note that we are considering correct detection with probability 1 (over the random choice of \mathbf{A}) for a *single* k -sparse \mathbf{x} . It is elementary to show that correct detection with probability 1 (again over the random choice of \mathbf{A}) for *all* k -sparse \mathbf{x} requires $m \geq 2k$.

B. ML Detection With Noise

Now suppose there is noise. Since \mathbf{x} is an unknown deterministic quantity, the probability of error in detecting the support is minimized by maximum likelihood (ML) detection. Since the noise \mathbf{d} is Gaussian, the ML detector finds the k -dimensional subspace spanned by k columns of \mathbf{A} containing the maximum energy of \mathbf{y} .

The ML estimator was first analyzed by Wainwright [3]. He shows that there exists a constant $C > 0$ such that if

$$\begin{aligned} m &\geq C \max \left\{ \frac{1}{\text{MAR} \cdot \text{SNR}} k \log(n-k), k \log(n/k) \right\} \\ &= C \max \left\{ \frac{1}{\text{SNR}_{\min}} \log(n-k), k \log(n/k) \right\} \end{aligned} \quad (10)$$

then ML will asymptotically detect the correct support. The equivalence of the two expressions in (10) is due to (8). Also, [5, Thm. 1] (generalized in [36, Thm. 1]) shows that, for any $\delta > 0$, the condition

$$\begin{aligned} m &\geq \frac{2(1-\delta)}{\text{MAR} \cdot \text{SNR}} k \log(n-k) + k \\ &= \frac{2(1-\delta)}{\text{SNR}_{\min}} \log(n-k) + k, \end{aligned} \quad (11)$$

is necessary. Observe that when $\text{SNR} \cdot \text{MAR} \rightarrow \infty$, the lower bound (11) approaches $m \geq k$, matching the noise-free case (9) as expected.

These necessary and sufficient conditions for ML appear in Table I with smaller terms and the infinitesimal δ omitted for simplicity.

C. Thresholding

The simplest method to detect the support is to use a thresholding rule of the form

$$\hat{I}_T = \{j \in \{1, 2, \dots, n\} : \rho(j) > \mu\}, \quad (12)$$

where $\mu > 0$ is a threshold parameter and $\rho(j)$ is the correlation coefficient:

$$\rho(j) = \frac{|\mathbf{a}'_j \mathbf{y}|^2}{\|\mathbf{a}_j\|^2 \|\mathbf{y}\|^2}, \quad j = 1, 2, \dots, n.$$

Thresholding has been analyzed in [5], [30], [37]. In particular, [5, Thm. 2] is the following: Suppose

$$\begin{aligned} m &> \frac{2(1+\delta)(1+\text{SNR})k L(k, n)}{\text{SNR} \cdot \text{MAR}} \\ &= \frac{2(1+\delta)(1+\text{SNR})L(k, n)}{\text{SNR}_{\min}} \end{aligned} \quad (13)$$

where $\delta > 0$ and

$$L(k, n) = [\sqrt{\log(n-k)} + \sqrt{\log(k)}]^2. \quad (14)$$

Then there exists a sequence of detection thresholds $\mu = \mu(n)$ such that \hat{I}_T achieves asymptotic reliable detection of the support. As before, the equivalence of the two expressions in (13) is due to (8).

Comparing the sufficient condition (13) for thresholding with the *necessary* condition (11), we see two distinct problems with thresholding:

- *Constant offset:* The scaling (13) for thresholding shows a factor $L(k, n)$ instead of $\log(n-k)$ in (11). It is easily verified that, for $k/n \in (0, 1/2)$,

$$\log(n-k) < L(k, n) < 4 \log(n-k), \quad (15)$$

so this difference in factors alone could require that thresholding use up to 4 times more measurements than ML for asymptotic reliable detection.

Combining the inequality (15) with (13), we see that the more stringent, but simpler, condition

$$m > \frac{8(1+\delta)(1+\text{SNR})}{\text{SNR} \cdot \text{MAR}} k \log(n-k) \quad (16)$$

is also sufficient for asymptotic reliable detection with thresholding. This simpler condition is shown in Table I, where we have omitted the infinitesimal δ quantity to simplify the table entry.

- *SNR saturation:* In addition to the $L(k, n)/\log(n-k)$ offset, thresholding also requires a factor of $1 + \text{SNR}$ more measurements than ML. This $1 + \text{SNR}$ factor has a natural interpretation as *intrinsic interference*: When detecting any one component of the vector \mathbf{x} , thresholding sees the energy from the other $n-1$ components of the signal as interference. This interference is distinct from the additive noise \mathbf{d} , and it increases the effective noise by a factor of $1 + \text{SNR}$.

The intrinsic interference results in a large performance gap at high SNRs. In particular, as $\text{SNR} \rightarrow \infty$, (13) reduces to

$$m > \frac{2(1+\delta)k L(k, n)}{\text{MAR}}. \quad (17)$$

In contrast, ML may be able to succeed with a scaling $m = O(k)$ for high SNRs.

D. Lasso and OMP Detection

While ML has clear advantages over thresholding, it is not computationally tractable for large problems. One practical method is *lasso* [14], also called basis pursuit denoising [13]. The lasso estimate of \mathbf{x} is obtained by solving the convex optimization

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} (\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \mu \|\mathbf{x}\|_1),$$

where $\mu > 0$ is an algorithm parameter that encourages sparsity in the solution $\hat{\mathbf{x}}$. The nonzero components of $\hat{\mathbf{x}}$ can then be used as an estimate of I_{true} .

Wainwright [4] has given necessary and sufficient conditions for asymptotic reliable detection with lasso. Partly because of freedom in the choice of a sequence of parameters $\mu(n)$, the finite SNR results are difficult to interpret. Under certain conditions with SNR growing unboundedly with n , matching necessary and sufficient conditions can be found. Specifically, if m , n and $k \rightarrow \infty$, with $\text{SNR} \cdot \text{MAR} \rightarrow \infty$, the scaling

$$m > 2k \log(n-k) + k + 1 \quad (18)$$

is both necessary and sufficient for asymptotic reliable detection.

Another common approach to support detection is the OMP algorithm [8]–[10]. This was analyzed by Tropp and Gilbert [28] in a setting with no noise. This was generalized to the present setting with noise by Fletcher and Rangan [29]. The result is very similar to condition (18): If m , n and $k \rightarrow \infty$, with

SNR · MAR $\rightarrow \infty$, a *sufficient* condition for asymptotic reliable recovery is

$$m > 2k \log(n - k). \quad (19)$$

The main result of [29] also allows uncertainty in k .

The conditions (18) and (19) are both shown in Table I. As usual, the table entries are simplified by including only the leading terms.

The lasso and OMP scaling laws, (18) and (19), can be compared with the high SNR limit for the thresholding scaling law in (17). This comparison shows the following:

- *Removal of the constant offset:* The $L(k, n)$ factor in the thresholding expression is replaced by a $\log(n - k)$ factor in the lasso and OMP scaling laws. Similar to the discussion above, this implies that lasso and OMP could require up to 4 times fewer measurements than thresholding.
- *Dynamic range:* In addition, both the lasso and OMP methods do not have a dependence on MAR. This gain can be large when there is high dynamic range, i.e., MAR is near zero.
- *Limits at high SNR:* We also see from (18) and (19) that both lasso and OMP are unable to achieve the scaling $m = O(k)$ that may be achievable with ML at high SNR. Instead, both lasso and OMP have the scaling $m = O(k \log(n - k))$, similar to the minimum scaling possible with thresholding.

E. Other Sparsity Detection Algorithms

Recent interest in compressed sensing has led to a plethora of algorithms beyond OMP and lasso. Empirical evidence suggests that the most promising algorithms for support detection are the sparse Bayesian learning methods developed in the machine learning community [38] and introduced into signal processing applications in [39], with related work in [40]. Unfortunately, a comprehensive summary of these algorithms is far beyond the scope of this paper. Our interest is not in finding the optimal algorithm, but rather to explain qualitative differences between algorithms and to demonstrate the value of knowing conditional ranks *a priori*.

IV. SEQUENTIAL ORTHOGONAL MATCHING PURSUIT

The results summarized in the previous section suggest a large performance gap between ML detection and practical algorithms such as thresholding, lasso and OMP, especially when the SNR is high. Specifically, as the SNR increases, the performance of these practical methods saturates at a scaling in the number of measurements that can be significantly higher than that for ML.

In this section, we introduce an OMP-like algorithm, which we call *sequential orthogonal matching pursuit*, that under favorable conditions can break this barrier. Specifically, in some cases, the performance of SequOMP does not saturate at high SNR.

A. Algorithm: SequOMP

Given a received vector \mathbf{y} , threshold level $\mu > 0$, and detection order π (a permutation on $\{1, 2, \dots, n\}$), the algorithm produces an estimate \hat{I}_S of the support I_{true} with the following steps:

- 1) Initialize the counter $j = 1$ and set the initial support estimate to empty: $\hat{I}(0) = \emptyset$.
- 2) Compute $\mathbf{P}(j)\mathbf{a}_{\pi(j)}$ where $\mathbf{P}(j)$ is the projection operator onto the orthogonal complement of the span of $\{\mathbf{a}_{\pi(\ell)}, \pi(\ell) \in \hat{I}(j - 1)\}$.
- 3) Compute the squared correlation between $\mathbf{P}(j)\mathbf{a}_{\pi(j)}$ and $\mathbf{P}(j)\mathbf{y}$:

$$\rho(j) = \frac{|\mathbf{a}'_{\pi(j)} \mathbf{P}(j)\mathbf{y}|^2}{\|\mathbf{P}(j)\mathbf{a}_{\pi(j)}\|^2 \|\mathbf{P}(j)\mathbf{y}\|^2}.$$

- 4) If $\rho(j) > \mu$, add the index $\pi(j)$ to $\hat{I}(j - 1)$. That is, $\hat{I}(j) = \hat{I}(j - 1) \cup \{j\}$. Otherwise, set $\hat{I}(j) = \hat{I}(j - 1)$.
- 5) Increment j to $j + 1$. If $j \leq n$ return to step 2.
- 6) The final estimate of the support is $\hat{I}_S = \hat{I}(n)$.

The SequOMP algorithm can be thought of as an iterative version of thresholding with the difference that, after a nonzero component is detected, subsequent correlations are performed only in the orthogonal complement to the corresponding column of \mathbf{A} . The method is identical to the standard OMP algorithm of [8]–[10], except that SequOMP passes through the data only once, in a fixed order. For this reason, SequOMP is computationally simpler than standard OMP.

As simulations will illustrate later, SequOMP generally has much worse performance than standard OMP. It is not intended as a competitive practical alternative. Our interest in the algorithm lies in the fact that we can prove positive results for SequOMP. Specifically, we will be able to show that this simple algorithm, when used in conjunction with known conditional ranks, can achieve a fundamentally better scaling at high SNRs than what has been proven is achievable with methods such as lasso and OMP.

B. Sequential OMP Performance

The analyses in Section III hold for deterministic vectors \mathbf{x} . Recall the partially-random signal model (2) where b_j is a Bernoulli(λ) random variable while the value of x_j conditional on x_j being nonzero remains deterministic; i.e., s_j is deterministic.

Let p_j denote the conditional energy of x_j , conditioned on $b_j = 1$ (i.e., $j \in I_{\text{true}}$). Then

$$p_j = s_j^2, \quad j = 1, 2, \dots, n. \quad (20)$$

We will call $\{p_j\}_{j=1}^n$ the *power profile*. Since $\Pr(b_j = 1) = \lambda$ for every j , the average value of $\text{SNR}(\mathbf{x})$ in (4) is given by

$$\text{SNR} = \mathbf{E}[\text{SNR}(\mathbf{x})] = \lambda \sum_{j=1}^n p_j. \quad (21)$$

Also, in analogy with $\text{MAR}(\mathbf{x})$ and $\text{SNR}_{\min}(\mathbf{x})$ in (6) and (7), define

$$\begin{aligned} \text{SNR}_{\min} &= \min_j p_j, \\ \text{MAR} &= \frac{\lambda n}{\text{SNR}} \min_j p_j = \frac{\lambda n \text{SNR}_{\min}}{\text{SNR}}. \end{aligned}$$

Note that the power profile p_j and the quantities SNR , SNR_{\min} and MAR as defined above are deterministic.

To simplify notation, we henceforth assume π is the identity permutation, i.e., the detection order in SequOMP is simply $(1, 2, \dots, n)$. A key parameter in analyzing the performance of

SequOMP is what we will call the *minimum signal-to-interference and noise ratio (MSINR)*

$$\gamma = \min_{\ell=1, \dots, n} p_\ell / \hat{\sigma}^2(\ell), \quad (22a)$$

where $\hat{\sigma}^2(\ell)$ is given by

$$\hat{\sigma}^2(\ell) = 1 + \lambda \sum_{j=\ell+1}^n p_j, \quad \ell = 1, 2, \dots, n. \quad (22b)$$

The parameters γ and $\hat{\sigma}^2(\ell)$ have simple interpretations: Suppose SequOMP has correctly detected b_j for all $j < \ell$. Then, in detecting b_ℓ , the algorithm sees the noise \mathbf{d} with power $\mathbb{E}[\|\mathbf{d}\|^2] = 1$ plus, for each component $j > \ell$, an interference power p_j with probability λ . Hence, $\hat{\sigma}^2(\ell)$ is the total average interference power seen when detecting b_ℓ , assuming perfect cancellation up to that point. Since the conditional power of x_ℓ is p_ℓ , the ratio $p_\ell / \hat{\sigma}^2(\ell)$ in (22a) represents the average SINR seen while detecting component ℓ . The value γ is the minimum SINR over all n components.

Theorem 1: Let $\lambda = \lambda(n)$, $m = m(n)$, and the power profile $\{p_j\}_{j=1}^n = \{p_j(n)\}_{j=1}^n$ be deterministic quantities varying with n that satisfy

$$\lim_{n \rightarrow \infty} \lambda n = \infty, \quad \lim_{n \rightarrow \infty} (1 - \lambda)n = \infty, \quad (23a)$$

$$\lim_{n \rightarrow \infty} \frac{m - \lambda n}{\log(\lambda n)} = \infty, \quad \lim_{n \rightarrow \infty} \frac{m - \lambda n}{\log((1 - \lambda)n)} = \infty. \quad (23b)$$

Also, assume the sequence of power profiles satisfies the limit

$$\lim_{n \rightarrow \infty} \max_{i=1, \dots, n-1} \log(n) \hat{\sigma}^{-4}(i) \sum_{j>i}^n p_j^2 = 0. \quad (23c)$$

Finally, assume that for all n ,

$$m \geq \frac{2(1 + \delta)(1 + \sqrt{\gamma})^2 L(\lambda n, n)}{\gamma} + \lambda n, \quad (24)$$

for some $\delta > 0$ where $L(\cdot, \cdot)$ is defined in (14) and γ is defined in (22a). Then, there exists a sequence of thresholds, $\mu = \mu(n)$, such that SequOMP with detection order $(1, 2, \dots, n)$ will achieve asymptotic reliable detection. The sequence of threshold levels can be selected independent of the sequence of power profiles.

Proof: See Appendix A. ■

The theorem provides a simple sufficient condition on the number of measurements as a function of the MSINR γ , probability λ , and dimension n . The condition (23c) is somewhat technical; we will verify its validity in examples. The remainder of this section discusses some of the implications of this theorem.

C. Most Favorable Detection Order With Known Conditional Ranks

Suppose that the *ordering* of the conditional power levels $\{p_j\}_{j=1}^n$ is known at the detector, but possibly not the values themselves. Reordering the power profile is equivalent to changing the detection order, so we seek the most favorable ordering of the power profile. Since $\hat{\sigma}^2(\ell)$ defined in (22b)

involves the sum of the tail of the power profile, the MSINR defined in (22a) is maximized when the power profile is non-increasing:

$$p_1 \geq p_2 \geq \dots \geq p_n = \text{SNR}_{\min}. \quad (25)$$

In other words, *the best detection order for SequOMP is from strongest component to weakest component.*

Using (25), it can be verified that the MSINR γ is bounded below by

$$\gamma \geq \frac{\text{SNR}_{\min}}{1 + \lambda n \text{SNR}_{\min}} = \frac{\text{SNR} \cdot \text{MAR}}{\lambda n (1 + \text{SNR} \cdot \text{MAR})}. \quad (26)$$

Furthermore, in cases of interest $\gamma \rightarrow 0$, so the sufficiency of the scaling (24) shows that

$$m \geq \frac{2(1 + \delta)\lambda n (1 + \text{SNR} \cdot \text{MAR})}{\text{SNR} \cdot \text{MAR}} L(\lambda n, n) + \lambda n \quad (27)$$

is sufficient for asymptotic reliable detection. This expression is shown in Table I with the additional simplification that $L(\lambda n, n) \leq 4 \log(n(1 - \lambda))$ for $\lambda \in (0, 1/2)$. To keep the notation consistent with the expressions for the other entries in the table, we have used k for λn , which is the average number of nonzero entries of \mathbf{x} .

When $\text{SNR} \rightarrow \infty$, (27) simplifies to

$$m \geq 2(1 + \delta)\lambda n L(\lambda n, n) + \lambda n. \quad (28)$$

This is identical to the lasso and OMP performance except for the factor $L(\lambda n, n) / \log((1 - \lambda)n)$, which lies in $(1, 4)$ for $\lambda \in (0, 1/2)$. In particular, the minimum number of measurements does not depend on MAR; therefore, similar to lasso and OMP, SequOMP can theoretically detect components that are much below the average power at high SNRs. More generally, we can say that knowledge of the conditional ranks of the powers enables a very simple algorithm to achieve resistance to large dynamic ranges.

D. Optimal Power Shaping

The MSINR lower bound in (26) is achieved as $n \rightarrow \infty$ and the power profile is constant (all p_j 's are equal). Thus, opposite to thresholding, a constant power profile is in some sense the *worst* power profile for a given SNR_{\min} for the SequOMP algorithm.

This raises the question: What is the most favorable power profile? Any power profile maximizing the MSINR γ subject to a constraint on total SNR (21) will achieve the minimum in (22a) for every ℓ and thus satisfy

$$p_\ell = \gamma \left(1 + \lambda \sum_{j=\ell+1}^n p_j \right), \quad \ell = 1, 2, \dots, n. \quad (29)$$

The solution to (29) and (21) is given by

$$p_\ell = \gamma_{\text{opt}} (1 + \gamma_{\text{opt}} \lambda)^{n-\ell}, \quad \ell = 1, 2, \dots, n, \quad (30a)$$

where

$$\gamma_{\text{opt}} = \frac{1}{\lambda} [(1 + \text{SNR})^{1/n} - 1] \approx \frac{1}{\lambda n} \log(1 + \text{SNR}) \quad (30b)$$

and the approximation holds for large n .² Again, some algebra shows that when λ is bounded away from zero, the power profile in (30) will satisfy the technical condition (23c) when $\log(1 + \text{SNR}) = o(n/\log(n))$.

The power profile (30a) is exponentially decreasing in the index order ℓ . Thus, components early in the detection sequence are allocated exponentially higher power than components later in the sequence. This allocation insures that early components have sufficient power to overcome the interference from all the components later in the detection sequence that are not yet cancelled.

Substituting (30b) into (24), we see that the scaling

$$m \geq \frac{2(1 + \delta)L(\lambda n, n)}{\log(1 + \text{SNR})} \lambda n + \lambda n \quad (31)$$

is sufficient for SequOMP to achieve asymptotic reliable detection with the best-case power profile. This expression is shown in Table I, again with the additional simplification that $L(\lambda n, n) \leq 4 \log(n(1 - \lambda))$ for $\lambda \in (0, 1/2)$.

E. SNR Saturation

As discussed earlier, a major problem with thresholding, lasso, and OMP is that their performances “saturate” with high SNR. That is, even as the SNR scales to infinity, the minimum number of measurements scales as $m = \Theta(\lambda n \log((1 - \lambda)n))$. In contrast, optimal ML detection can achieve a scaling $m = O(\lambda n)$, when the SNR is sufficiently high.

A consequence of (31) is that SequOMP with exponential power shaping can overcome this barrier. Specifically, if we take the scaling of $\text{SNR} = \Theta(\lambda n)$ in (31), apply the bound $L(\lambda n, n) \leq 4 \log(n(1 - \lambda))$ for $\lambda \in (0, 1/2)$, and assume that λ is bounded away from zero, we see that asymptotically, SequOMP requires only $m \geq 9\lambda n$ measurements. In this way, unlike thresholding and lasso, SequOMP is able to succeed with scaling $m = O(\lambda n)$ when $\text{SNR} \rightarrow \infty$. In fact, if SNR grows slightly faster so that it satisfies $L(\lambda n, n) = o(\log(1 + \text{SNR}))$ while still satisfying $\log(1 + \text{SNR}) = o(n/\log(n))$, then (31) leads to an asymptotic sufficient condition of $m > n$.

F. Power Shaping With Sparse Bayesian Learning

The fact that power shaping can provide benefits when combined with certain iterative detection algorithms confirms the observations in the work of Wipf and Rao [41]. That work considers signal detection with a certain sparse Bayesian learning (SBL) algorithm. They show the following result: Suppose \mathbf{x} has k nonzero components and p_i , $i = 1, 2, \dots, k$, is the power of the i th largest component. Then, for a given measurement matrix \mathbf{A} , there exist constants $\nu_i > 1$ such that if

$$p_i \geq \nu_i p_{i-1}, \quad i = 2, 3, \dots, k, \quad (32)$$

the SBL algorithm will correctly detect the sparsity pattern of \mathbf{x} .

²The solution (30) is the $\theta = 0$ case of a more general result in Section IV.G; see (35).

The condition (32) shows that a certain growth in the powers can guarantee correct detection. The parameters ν_i however depend in some complex manner on the matrix \mathbf{A} , so the appropriate growth is difficult to compute. They also provide strong empirical evidence that shaping the power with certain profiles can greatly reduce the number of measurements needed.

The results in this paper add to Wipf and Rao’s observations showing that growth in the powers can also assist SequOMP. Moreover, for SequOMP, we can explicitly derive the optimal power profile for certain large random matrices.

This is not to say that SequOMP is better than SBL. In fact, empirical results in [39] suggest that SBL will outperform OMP, which will in turn do better than SequOMP. As we have stressed before, the point of analyzing SequOMP here is that we can derive concrete analytic results. These results may provide guidance for more sophisticated algorithms.

G. Robust Power Shaping

The above analysis shows certain benefits of SequOMP used in conjunction with power shaping. The results are proven for reliable detection of *all* entries of the support in a limit of unbounded block length (see Definition 1). In problems of finite size or at operating points where a nonzero fraction of errors is tolerable, the power shaping above may hurt performance.

When a nonzero component is not detected in SequOMP, that component’s energy is not cancelled out and remains as interference for all subsequent components in the detection sequence. With power shaping, components early in the detection sequence have much higher power than components later in the sequence. Compared to a case with the same SNR and a constant power profile, the use of power shaping reduces the probability of an early missed detection but increases the harm in subsequent steps that comes from such a missed detection. As block length increases, the probability of missed detection can be driven to zero. But at any finite block length, the probability of a missed detection early in the sequence will always be nonzero.

The work [42] observed a similar problem when successive interference cancellation is used in a CDMA uplink. To mitigate the problem, [42] proposed to adjust the power allocations to make them more robust to detection errors early in the detection sequence. The same technique, which we will call *robust power shaping*, can be applied to SequOMP as follows.

The condition (29) is motivated by maintaining a constant MSINR through the detection process, assuming all components with indexes $j < \ell$ have been correctly detected and subtracted. An alternative, following [42], is to assume that some fixed fraction $\theta \in [0, 1]$ of the energy of components early in the detection sequence is not cancelled out due to missed detections. We will call θ the *leakage fraction*. With nonzero leakage, the condition (29) is replaced by

$$p_\ell = \gamma \left(1 + \theta \lambda \sum_{j=1}^{\ell-1} p_j + \lambda \sum_{j=\ell+1}^n p_j \right), \quad \ell = 1, 2, \dots, n. \quad (33)$$

For given γ , λ , and θ , (33) in a system of linear equations that determines the power profile $\{p_\ell\}_{\ell=1}^n$; one can vary γ until the power profile provides the desired SNR according to (21).

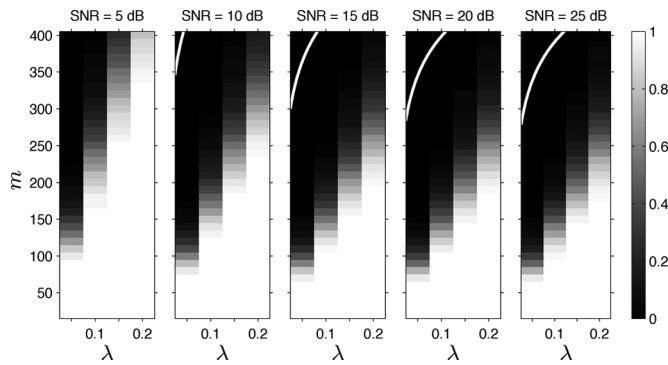


Fig. 1. Probability of error in full support recovery with SequOMP for a signal with $n = 100$ components when the power profile is optimized as in (35) with leakage fraction $\theta = 0.1$. Each shaded box presents the result of 2000 Monte Carlo trials with $\lambda \in \{0.05, 0.1, 0.15, 0.2\}$, $m \in \{20, 40, \dots, 400\}$ and SNR as indicated. The white line shows the theoretical sufficient condition on m obtained from Theorem 1.

A closed-form solution to (33) provides some additional insight. Adding and subtracting SNR inside the parentheses in (33) while also using (21) yields

$$p_\ell = \gamma \left(\underbrace{1 + \text{SNR} - \lambda \sum_{j=1}^n p_j}_{=0} + \theta \lambda \sum_{j=1}^{\ell-1} p_j + \lambda \sum_{j=\ell+1}^n p_j \right),$$

which can be rearranged to

$$(1 + \gamma \lambda) p_\ell = \gamma \left(1 + \text{SNR} - (1 - \theta) \lambda \sum_{j=1}^{\ell-1} p_j \right). \quad (34)$$

Using standard techniques for solving linear constant-coefficient difference equations,

$$p_j = \frac{\text{SNR}}{\lambda} \cdot \frac{(1 - \zeta) \zeta^{j-1}}{1 - \zeta^n} \quad (35a)$$

where

$$\zeta = \frac{1 + \gamma \lambda \theta}{1 + \gamma \lambda} \quad (35b)$$

and

$$\gamma = \frac{1}{\lambda} \cdot \frac{1 - \left(\frac{1 + \theta \text{SNR}}{1 + \text{SNR}} \right)^{1/n}}{\left(\frac{1 + \theta \text{SNR}}{1 + \text{SNR}} \right)^{1/n} - \theta}. \quad (35c)$$

Notice that $\theta < 1$ implies $\zeta < 1$, so the power profile (35a) is decreasing as in the case without leakage in Section IV-D. Setting $\theta = 0$ recovers (30).

V. NUMERICAL SIMULATIONS

A. Basic Validation of Sufficient Condition

We first compare the actual performance of the SequOMP algorithm with the sufficient condition for support recovery in Theorem 1. Fig. 1 shows the simulated probability of error p_{err} obtained using SequOMP at various SNR levels, probabilities of

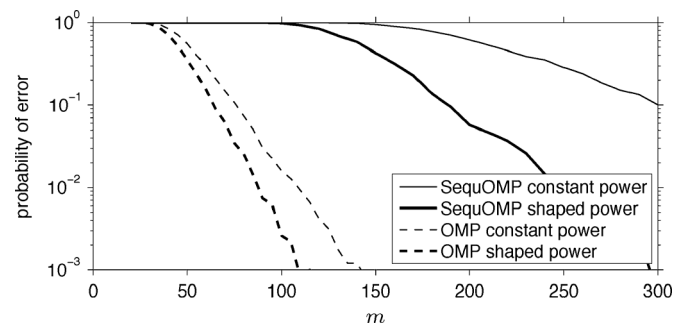


Fig. 2. Comparison of SequOMP and OMP, with and without power shaping. Probability of error in full support recovery is plotted as a function of m with number of components $n = 100$, probability of an entry being nonzero $\lambda = 0.1$, and SNR = 25 dB. The power profile is either constant or optimized as in (35) with leakage fraction $\theta = 0.1$.

nonzero components λ , and numbers of measurements m . In all these simulations, the number of components was fixed to $n = 100$, and each shaded box represents an empirical probability of error over 2000 independent Monte Carlo trials. The robust power profile of Section IV-G is used with a leakage fraction $\theta = 0.1$. Here and in subsequent simulations, the threshold μ is set to the level specified in (39) in the proof of Theorem 1 in Appendix A.³

The white line in Fig. 1 represents the number of measurements m for which Theorem 1 would theoretically guarantee reliable detection of the support at infinite block lengths. To apply the theorem, we used the MSINR γ from (35c). At the block lengths considered in these simulations, the probability of error at the theoretical sufficient condition is small, typically under 0.1%. The theoretical sufficient condition shows the same trends as the empirical results.

B. Effect of Power Shaping

Fig. 2 compares the performances of SequOMP and OMP, with and without power shaping. In the simulations, $n = 100$, $\lambda = 0.1$, and the total SNR is 25 dB. When power shaping is used, the power profile is determined through (35) with leakage fraction $\theta = 0.1$. Otherwise, the power profile is constant. The number of measurements m was varied, and for each m , the probability of error was estimated with 5000 independent Monte Carlo trials.

As expected from the theoretical analysis in this paper, with the total SNR kept constant, the performance of SequOMP is improved by optimization of the power profile. As also is to be expected, SequOMP is considerably worse than OMP in terms of error probability for a given number of measurements or number of measurements needed to achieve a given error probability. Our interest in SequOMP is that it is amenable to analysis; OMP presumably performs better than SequOMP in any setting of interest, but it does not do so for every problem instance, so our analysis does not carry over to OMP rigorously.

The simulation in Fig. 2 shows that power shaping provides gains with OMP as well. As discussed in Section IV-F, this is consistent with observations in the work of Wipf and Rao [41].

³Simulations presented in [43] use a different choice of μ . There, μ is adjusted to achieve a fixed false alarm probability and all plotted quantities are missed detection probabilities. The conclusions are qualitatively similar.

VI. CONCLUSION

Methods such as OMP and lasso, which are widely used in sparse signal support detection problems, exhibit advantages over thresholding but still fall far short of the performance of optimal (ML) detection at high SNRs. Analysis of the SequOMP algorithm has shown that knowledge of conditional rank of signal components enables performance similar to OMP and lasso at a lower complexity. Furthermore, in the most favorable situations, conditional rank knowledge changes the fundamental scaling of performance with SNR so that performance no longer saturates with SNR.

APPENDIX
PROOF OF THEOREM 1

A. Proof Outline

At a high level, the proof of Theorem 1 is similar to the proof of [5, Thm. 2], the thresholding condition (16). One of the difficulties in the proof is to handle the dependence between random events at different iterations of the SequOMP algorithm. To avoid this difficulty, we first show an equivalence between the success of SequOMP and an alternative sequence of events that is easier to analyze. After this simplification, small modifications handle the cancellations of detected vectors.

Fix n and define $I_{\text{true}}(j) = \{\ell : \ell \in I_{\text{true}}, \ell \leq j\}$, which is the set of elements of the true support with indices $\ell \leq j$. Observe that $I_{\text{true}}(0) = \emptyset$ and $I_{\text{true}}(n) = I_{\text{true}}$.

Let $\mathbf{P}_{\text{true}}(j)$ be the projection operator onto the orthogonal complement of $\{\mathbf{a}_\ell, \ell \in I_{\text{true}}(j-1)\}$, and define

$$\rho_{\text{true}}(j) = \frac{|\mathbf{a}'_j \mathbf{P}_{\text{true}}(j) \mathbf{y}|^2}{\|\mathbf{P}_{\text{true}}(j) \mathbf{a}_j\|^2 \|\mathbf{P}_{\text{true}}(j) \mathbf{y}\|^2}. \quad (36)$$

A simple induction argument shows that SequOMP correctly detects the support if and only if, at each iteration j , the variables $\hat{I}(j)$, $\mathbf{P}(j)$ and $\rho(j)$ defined in the algorithm are equal to $I_{\text{true}}(j)$, $\mathbf{P}_{\text{true}}(j)$ and $\rho_{\text{true}}(j)$, respectively. Therefore, if we define $\hat{I} = \{j : \rho_{\text{true}}(j) > \mu\}$, then SequOMP correctly detects the support if and only if $\hat{I} = I_{\text{true}}$. In particular, $p_{\text{err}}(n) = \Pr(\hat{I} \neq I_{\text{true}})$.

To prove that $p_{\text{err}}(n) \rightarrow 0$ it suffices to show that there exists a sequence of threshold levels $\mu(n)$ such that

$$\liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}(n)} \frac{\rho_{\text{true}}(j)}{\mu} > 1, \quad (37a)$$

$$\limsup_{n \rightarrow \infty} \max_{j \notin I_{\text{true}}(n)} \frac{\rho_{\text{true}}(j)}{\mu} < 1, \quad (37b)$$

hold in probability. The first limit (37a) ensures that all the components in the true support will not be missed and will be called the *zero missed detection condition*. The second limit (37b) ensures that all the components not in the true support will not be falsely detected and will be called the *zero false alarm condition*.

Set the sequence of threshold levels as follows. Since $\delta > 0$, we can find an $\epsilon > 0$ such that

$$(1 + \delta) \geq (1 + \epsilon)^2. \quad (38)$$

For each n , let the threshold level be

$$\mu = (1 + \epsilon) (1 + \sqrt{\gamma})^2 \frac{2 \log(n(1 - \lambda))}{m - \lambda n}. \quad (39)$$

The asymptotic lack of missed detections and false alarms with these thresholds are proven in Appendices D and E, respectively. In preparation for these sections, Appendix B reviews some facts concerning tail bounds on chi-squared and beta random variables and Appendix C presents some preliminary computations.

B. Chi-Squared and Beta Random Variables

The proof requires a number of simple facts concerning chi-squared and beta random variables. These variables are reviewed in [44]. We omit all the proofs in this subsection and instead reference very closely related lemmas in [5].

A random variable u has a *chi-squared* distribution with r degrees of freedom if it can be written as $u = \sum_{i=1}^r z_i^2$, where z_i are i.i.d. $\mathcal{N}(0, 1)$.

Lemma 1 ([5], Lemma 2): Suppose $\mathbf{x} \in \mathbb{R}^r$ has a Gaussian distribution $\mathcal{N}(0, \sigma^2 I_r)$. Then:

- $\|\mathbf{x}\|^2 / \sigma^2$ is chi-squared with r degrees of freedom; and
- if \mathbf{y} is any other r -dimensional random vector that is nonzero with probability one and independent of \mathbf{x} , then the variable $u = |\mathbf{x}'\mathbf{y}|^2 / (\sigma^2 \|\mathbf{y}\|^2)$ is a chi-squared random variable with one degree of freedom.

The following two lemmas provide standard tail bounds.

Lemma 2 (Similar to [5], Lemma 3): Suppose that for each n , $\{\mathbf{x}_j^{(n)}\}_{j=1}^n$ is a set of Gaussian random vectors with each $\mathbf{x}_j^{(n)}$ spherically symmetric in an $m_j(n)$ -dimensional space. The variables may be dependent. Suppose also that $\mathbf{E}\|\mathbf{x}_j^{(n)}\|^2 = 1$ and $\lim_{n \rightarrow \infty} \log(n) / m_{\min}(n) = 0$ where $m_{\min}(n) = \min_{j=1, \dots, n} m_j(n)$. Then the limits

$$\lim_{n \rightarrow \infty} \max_{j=1, \dots, n} \left\| \mathbf{x}_j^{(n)} \right\|^2 = \lim_{n \rightarrow \infty} \min_{j=1, \dots, n} \left\| \mathbf{x}_j^{(n)} \right\|^2 = 1$$

hold in probability.

Lemma 3 ([5], Lemma 4): Suppose that for each n , $\{u_j^{(n)}\}_{j=1}^n$ is a set of chi-squared random variables, each with one degree of freedom. The variables may be dependent. Then

$$\limsup_{n \rightarrow \infty} \max_{j=1, \dots, n} \frac{u_j^{(n)}}{2 \log(n)} \leq 1,$$

where the limit is in probability.

The final two lemmas concern certain beta-distributed random variables. A real-valued scalar random variable w follows a $\text{Beta}(r, s)$ distribution if it can be written as $w = u_r / (u_r + v_s)$, where the variables u_r and v_s are independent chi-squared random variables with r and s degrees of freedom, respectively. The importance of the beta distribution is given by the following lemma.

Lemma 4 ([5], Lemma 5): Suppose \mathbf{x} and \mathbf{y} are independent random r -dimensional random vectors with \mathbf{x} being spherically-symmetrically distributed in \mathbb{R}^r and \mathbf{y} having any distribution that is nonzero with probability one. Then the random variable $w = |\mathbf{x}'\mathbf{y}|^2 / (\|\mathbf{x}\|^2 \|\mathbf{y}\|^2)$ is independent of \mathbf{x} and follows a $\text{Beta}(1, r-1)$ distribution.

The following lemma provides a simple expression for the maxima of certain beta-distributed variables.

Lemma 5 ([5, Lemma 6]): For each n , suppose $\{w_j^{(n)}\}_{j=1}^n$ is a set of random variables with $w_j^{(n)}$ having a Beta($1, m_j(n) - 1$) distribution. Suppose that

$$\lim_{n \rightarrow \infty} \log(n)/m_{\min}(n) = 0, \quad \lim_{n \rightarrow \infty} m_{\min}(n) = \infty,$$

where $m_{\min}(n) = \min_{j=1, \dots, n} m_j(n)$. Then,

$$\limsup_{n \rightarrow \infty} \max_{j=1, \dots, n} \frac{m_j(n)}{2 \log(n)} w_j^{(n)} \leq 1$$

in probability.

C. Preliminary Computations and Technical Lemmas

We first need to prove several simple but technical bounds. We begin by considering the dimension m_i defined as

$$m_i = \dim(\text{range}(\mathbf{P}_{\text{true}}(i))). \quad (40)$$

Our first lemma computes the limit of this dimension.

Lemma 6: The limit

$$\lim_{n \rightarrow \infty} \min_{i=1, \dots, n} \frac{m_i}{m - \lambda n} = 1 \quad (41)$$

holds in probability and almost surely.

Proof: Recall that $\mathbf{P}_{\text{true}}(i)$ is the projection onto the orthogonal complement of the vectors \mathbf{a}_j with $j \in I_{\text{true}}(i - 1)$. With probability one, these vectors will be linearly independent, so $\mathbf{P}_{\text{true}}(i)$ will have dimension $m - |I_{\text{true}}(i - 1)|$. Since $I_{\text{true}}(i)$ is increasing with i ,

$$\begin{aligned} \min_{i=1, \dots, n} m_i &= m - \max_{i=1, \dots, n} |I_{\text{true}}(i - 1)| \\ &= m - |I_{\text{true}}(n - 1)|. \end{aligned} \quad (42)$$

Since each index is in the support with probability λ and the b_j s are independent, the law of large numbers shows that

$$\lim_{n \rightarrow \infty} \frac{|I_{\text{true}}(n - 1)|}{\lambda(n - 1)} = 1$$

in probability and almost surely. Combining this with (42) and (23b) shows (41). \blacksquare

Next, for each $i = 1, \dots, n$, define the *residual vector*,

$$\mathbf{e}_i = \mathbf{P}_{\text{true}}(i)(\mathbf{y} - \mathbf{a}_i x_i). \quad (43)$$

Observe that

$$\begin{aligned} \mathbf{e}_i &= \mathbf{P}_{\text{true}}(i)(\mathbf{y} - \mathbf{a}_i x_i) \stackrel{(a)}{=} \mathbf{P}_{\text{true}}(i) \left(\mathbf{d} + \sum_{j \neq i} \mathbf{a}_j x_j \right) \\ &\stackrel{(b)}{=} \mathbf{P}_{\text{true}}(i) \left(\mathbf{d} + \sum_{j > i} \mathbf{a}_j x_j \right), \end{aligned}$$

where (a) follows from (1) and (b) follows from the fact that $\mathbf{P}_{\text{true}}(i)$ is the projection onto the orthogonal complement of the span of all vectors \mathbf{a}_j with $j < i$ and $x_j \neq 0$.

The next lemma shows that the power of the residual vector is described by the random variable

$$\sigma^2(i) = 1 + \sum_{j=i+1}^n |x_j|^2. \quad (44)$$

Lemma 7: For all $i = 1, \dots, n$, the residual vector \mathbf{e}_i , conditioned on the modulation vector \mathbf{x} and projection $\mathbf{P}_{\text{true}}(i)$, is a

spherically symmetric Gaussian in the range space of $\mathbf{P}_{\text{true}}(i)$ with total variance

$$\mathbf{E}(\|\mathbf{e}_i\|^2 | \mathbf{x}) = \frac{m_i}{m} \sigma^2(i), \quad (45)$$

where m_i and $\sigma^2(i)$ are defined in (40) and (44), respectively.

Proof: Let $\mathbf{v}_i = \mathbf{d} + \sum_{j>i} \mathbf{a}_j x_j$, so that $\mathbf{e}_i = \mathbf{P}_{\text{true}}(i) \mathbf{v}_i$. Since the vectors \mathbf{a}_j and \mathbf{d} have Gaussian $\mathcal{N}(0, 1/mI_m)$ distributions, for a given vector \mathbf{x} , \mathbf{v}_i must be a zero-mean white Gaussian vector with total variance $\mathbf{E}\|\mathbf{v}_i\|^2 = \sigma^2(i)$. Also, since the operator $\mathbf{P}_{\text{true}}(i)$ is a function of the components x_ℓ and vectors \mathbf{a}_ℓ for $\ell < i$, $\mathbf{P}_{\text{true}}(i)$ is independent of the vectors \mathbf{d} and \mathbf{a}_j , $j > i$, and therefore independent of \mathbf{v}_i . Since $\mathbf{P}_{\text{true}}(i)$ is a projection from an m -dimensional space to an m_i -dimensional space, \mathbf{e}_i , conditioned on the modulation vector \mathbf{x} , must be spherically symmetric Gaussian in the range space of $\mathbf{P}_{\text{true}}(i)$ with total variance satisfying (45). \blacksquare

Our next lemma requires the following version of the well-known Hoeffding's inequality.

Lemma 8 (Hoeffding's Inequality): Suppose z is the sum $z = z_0 + \sum_{i=1}^r z_i$ where z_0 is a constant and the variables z_i are independent random variables that are almost surely bounded in some interval $z_i \in [a_i, b_i]$. Then, for all $\epsilon > 0$,

$$\Pr(z - \mathbf{E}(z) \geq \epsilon) \leq \exp\left(\frac{-2\epsilon^2}{C}\right),$$

where $C = \sum_{i=1}^r (b_i - a_i)^2$.

Proof: See [45]. \blacksquare

Lemma 9: Under the assumptions of Theorem 1, the limit

$$\limsup_{n \rightarrow \infty} \max_{i=1, \dots, n} \frac{\sigma^2(i)}{\hat{\sigma}^2(i)} \leq 1$$

holds in probability.

Proof: Let $z(i) = \sigma^2(i)/\hat{\sigma}^2(i)$. From the definition of $\sigma^2(i)$ in (44), we can write

$$z(i) = \frac{1}{\hat{\sigma}^2(i)} + \sum_{j=i+1}^n z(i, j),$$

where $z(i, j) = |x_j|^2/\hat{\sigma}^2(i)$ for $j > i$.

Now recall that in the problem formulation, each x_j is nonzero with probability λ , with conditional power p_j . Also, the activity variables $\{b_j\}_{j=1}^n$ are independent, and the conditional powers p_j are deterministic quantities. Therefore, the variables $z(i, j)$ are independent with

$$z(i, j) = \begin{cases} p_j/\hat{\sigma}^2(i), & \text{with probability } \lambda; \\ 0, & \text{with probability } 1 - \lambda, \end{cases}$$

for $j > i$. Combining this with the definition of $\hat{\sigma}^2(i)$ in (22b), we see that

$$\mathbf{E}(z(i)) = \frac{1}{\hat{\sigma}^2(i)} \left(1 + \lambda \sum_{j=i+1}^n p_j \right) = 1.$$

Also, for each $j > i$, we have the bound $z(i, j) \in [0, p_j/\hat{\sigma}^2(i)]$. So for use in Hoeffding's Inequality (Lemma 8), define

$$C = C(i, n) = \hat{\sigma}^{-4}(i) \sum_{j=i+1}^n p_j^2,$$

where dependence of the power profile and $\hat{\sigma}(i)$ on n is implicit. Now define

$$c_n = \max_{i=1,\dots,n} \log(n)C(i, n),$$

so that $C(i, n) \leq c_n/\log(n)$ for all i . Hoeffding's Inequality (Lemma 8) now shows that for all $i < n$,

$$\begin{aligned} \Pr(z(i) \geq 1 + \epsilon) &\leq \exp(-2\epsilon^2/C(i, n)) \\ &\leq \exp(-2\epsilon^2 \log(n)/c_n). \end{aligned}$$

Using the union bound,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr\left(\max_{j=1,\dots,n} z(j) > 1 + \epsilon\right) \\ &\leq \lim_{n \rightarrow \infty} n \exp\left(-\frac{2\epsilon^2 \log(n)}{c_n}\right) \\ &= \lim_{n \rightarrow \infty} n^{1-2\epsilon^2/c_n} = 0. \end{aligned}$$

The final step is due to the fact that the technical condition (23c) in the theorem implies $c_n \rightarrow 0$. This proves the lemma. ■

D. Missed Detection Probability

Consider any $j \in I_{\text{true}}$. Using (43) to rewrite (36) along with some algebra shows

$$\begin{aligned} \rho_{\text{true}}(j) &= \frac{|\mathbf{a}'_j \mathbf{P}_{\text{true}}(j) \mathbf{y}|^2}{\|\mathbf{P}_{\text{true}}(j) \mathbf{a}\|^2 \|\mathbf{P}_{\text{true}}(j) \mathbf{y}_j\|^2} \\ &= \frac{|\mathbf{a}'_j (x_j \mathbf{P}_{\text{true}}(j) \mathbf{a}_j + \mathbf{e}_j)|^2}{\|\mathbf{P}_{\text{true}}(j) \mathbf{a}_j\|^2 \|x_j \mathbf{P}_{\text{true}}(j) \mathbf{a}_j + \mathbf{e}_j\|^2} \\ &\geq \frac{s_j - 2\sqrt{z_j s_j} + z_j}{s_j + 2\sqrt{z_j s_j} + 1}, \end{aligned} \quad (46)$$

where

$$s_j = \frac{|x_j|^2 \|\mathbf{P}_{\text{true}}(j) \mathbf{a}_j\|^2}{\|\mathbf{e}_j\|^2}, \quad (47)$$

$$z_j = \frac{|\mathbf{a}'_j \mathbf{P}_{\text{true}}(j) \mathbf{e}_j|^2}{\|\mathbf{P}_{\text{true}}(j) \mathbf{a}_j\|^2 \|\mathbf{e}_j\|^2}. \quad (48)$$

Define

$$s_{\min} = \min_{j \in I_{\text{true}}} s_j, \quad s_{\max} = \max_{j \in I_{\text{true}}} z_j.$$

We will now bound s_{\min} from below and s_{\max} from above.

We first start with s_{\min} . Conditional on \mathbf{x} and $\mathbf{P}_{\text{true}}(j)$, Lemma 7 shows that each \mathbf{e}_j is a spherically-symmetrically distributed Gaussian on the m_j -dimensional range space of $\mathbf{P}_{\text{true}}(j)$. Since there are asymptotically λn elements in I_{true} , Lemma 2 along with (23b) show that

$$\lim_{n \rightarrow \infty} \max_{j \in I_{\text{true}}} \frac{m}{m_j \sigma^2(j)} \|\mathbf{e}_j\|^2 = 1, \quad (49)$$

where the limit is in probability. Similarly, $\mathbf{P}_{\text{true}}(j) \mathbf{a}_j$ is also a spherically-symmetrically distributed Gaussian in the range space of $\mathbf{P}_{\text{true}}(j)$. Since $\mathbf{P}_{\text{true}}(j)$ is a projection from an m -dimensional space to a m_j -dimensional space and $\mathbf{E}\|\mathbf{a}_j\|^2 = 1$,

we have that $\mathbf{E}\|\mathbf{P}_{\text{true}}(j) \mathbf{a}_j\|^2 = m_j/m$. Therefore, Lemma 2 along with (23b) show that

$$\lim_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{m}{m_j} \|\mathbf{P}_{\text{true}}(j) \mathbf{e}_j\|^2 = 1. \quad (50)$$

Taking the limit (in probability) of s_{\min} ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{s_{\min}}{\gamma} &= \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{s_j}{\gamma} \\ &\stackrel{(a)}{=} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{|x_j|^2 \|\mathbf{P}_{\text{true}}(j) \mathbf{a}_j\|^2}{\gamma \|\mathbf{e}_j\|^2} \\ &\stackrel{(b)}{=} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{|x_j|^2}{\gamma \sigma^2(j)} \\ &\stackrel{(c)}{=} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{p_j}{\gamma \sigma^2(j)} \\ &\stackrel{(d)}{\geq} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{p_j}{\gamma \hat{\sigma}^2(j)} \stackrel{(e)}{\geq} 1, \end{aligned} \quad (51)$$

where (a) follows from (47); (b) follows from (49) and (50); (c) follows from (20); (d) follows from Lemma 9; and (e) follows from (22a).

We next consider s_{\max} . Conditional on $\mathbf{P}_{\text{true}}(j)$, the vectors $\mathbf{P}_{\text{true}}(j) \mathbf{a}_j$ and \mathbf{e}_j are independent spherically-symmetric Gaussians in the range space of $\mathbf{P}_{\text{true}}(j)$. It follows from Lemma 4 that each z_j is a Beta(1, $m_j - 1$) random variable. Since there are asymptotically λn elements in I_{true} , Lemma 5 along with (41) and (23b) show that

$$\limsup_{n \rightarrow \infty} \frac{m - \lambda n}{2 \log(\lambda n)} s_{\max} = \limsup_{n \rightarrow \infty} \frac{m - \lambda n}{2 \log(\lambda n)} \max_{j \in I_{\text{true}}} z_j \leq 1. \quad (52)$$

The above analysis shows that for any $j \in I_{\text{true}}$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{1}{\sqrt{\mu}} (\sqrt{s_j} - \sqrt{z_j}) \\ &\stackrel{(a)}{\geq} \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{\mu}} (\sqrt{s_{\min}} - \sqrt{s_{\max}}) \\ &\stackrel{(b)}{\geq} \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{\mu}} \left(\sqrt{\gamma} - \sqrt{\frac{2 \log(\lambda n)}{m - \lambda n}} \right) \\ &\geq \liminf_{n \rightarrow \infty} \sqrt{\frac{1 + \delta}{\mu}} \left(\sqrt{\frac{\gamma}{1 + \delta}} - \sqrt{\frac{2 \log(\lambda n)}{m - \lambda n}} \right) \\ &\stackrel{(c)}{\geq} \liminf_{n \rightarrow \infty} \sqrt{\frac{2(1 + \delta)}{(m - \lambda n)\mu}} \\ &\quad \cdot ((1 + \sqrt{\gamma}) \sqrt{L(\lambda n, n)} - \sqrt{\log(\lambda n)}) \\ &\stackrel{(d)}{=} \liminf_{n \rightarrow \infty} (1 + \sqrt{\gamma}) \sqrt{\frac{2(1 + \delta) \log(n(1 - \lambda))}{(m - \lambda n)\mu}} \\ &\stackrel{(e)}{=} \liminf_{n \rightarrow \infty} (1 + \sqrt{\gamma}) \sqrt{\frac{1 + \delta}{1 + \epsilon}} \\ &\stackrel{(f)}{\geq} (1 + \sqrt{\gamma}) \sqrt{1 + \epsilon} \end{aligned} \quad (53)$$

where (a) follows from the definitions of s_{\min} and s_{\max} ; (b) follows from (51) and (52); (c) follows from (24); (d) follows from

(14); (e) follows from (39); and (f) follows from (38). Therefore, starting with (46),

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{\rho(j)}{\mu} \\
& \stackrel{(a)}{\geq} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{1}{\mu} \cdot \frac{s_j - 2\sqrt{z_j s_j} + z_j}{s_j + 2\sqrt{z_j s_j} + 1} \\
& = \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{1}{\mu} \cdot \frac{(\sqrt{s_j} - \sqrt{z_j})^2}{s_j + 2\sqrt{z_j s_j} + 1} \\
& \stackrel{(b)}{\geq} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{(1 + \sqrt{\gamma})^2(1 + \epsilon)}{s_j + 2\sqrt{z_j s_j} + 1} \\
& \stackrel{(c)}{\geq} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{(1 + \sqrt{\gamma})^2(1 + \epsilon)}{s_j + 2\sqrt{s_j} + 1} \\
& \geq \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{(1 + \sqrt{\gamma})^2(1 + \epsilon)}{s_{\min} + 2\sqrt{s_{\min}} + 1} \\
& \stackrel{(d)}{\geq} \liminf_{n \rightarrow \infty} \min_{j \in I_{\text{true}}} \frac{(1 + \sqrt{\gamma})^2(1 + \epsilon)}{(1 + \sqrt{\gamma})^2} = 1 + \epsilon,
\end{aligned}$$

where (a) follows from (46); (b) follows from (53); (c) follows from the fact that $z_j \in [0, 1]$ (it is a beta-distributed random variable); (d) follows from (51). This proves the first requirement, condition (37a).

E. False Alarm Probability

Now consider any index $j \notin I_{\text{true}}$. This implies that $x_j = 0$ and therefore (43) shows that

$$\mathbf{P}_{\text{true}}(j)\mathbf{y} = \mathbf{e}_j.$$

Hence from (36),

$$\rho_{\text{true}}(j) = \frac{|\mathbf{a}'_j \mathbf{e}_j|^2}{\|\mathbf{P}_{\text{true}}(j)\mathbf{a}\|^2 \|\mathbf{e}_j\|^2} = z_j \quad (54)$$

where z_j is defined in (48). From the discussion above, each z_j has the Beta(1, $m_j - 1$) distribution. Since there are asymptotically $(1 - \lambda)n$ elements in I_{true}^c , the conditions (41) and (23b) along with Lemma 5 show that the limit

$$\limsup_{n \rightarrow \infty} \max_{j \notin I_{\text{true}}} \frac{m - \lambda n}{2 \log(n(1 - \lambda))} z_j \leq 1 \quad (55)$$

holds in probability. Therefore,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \max_{j \notin I_{\text{true}}} \frac{1}{\mu} \rho_{\text{true}}(j) \\
& \stackrel{(a)}{=} \limsup_{n \rightarrow \infty} \max_{j \notin I_{\text{true}}} \frac{1}{\mu} z_j \\
& \stackrel{(b)}{=} \limsup_{n \rightarrow \infty} \max_{j \notin I_{\text{true}}} \frac{m - \lambda n}{(1 + \epsilon)(1 + \sqrt{\gamma})^2 2 \log(n(1 - \lambda))} z_j \\
& \stackrel{(c)}{\leq} \frac{1}{1 + \epsilon}
\end{aligned}$$

where (a) follows from (54); (b) follows from (39); and (c) follows from (55). This proves (37b) and thus completes the proof of the theorem.

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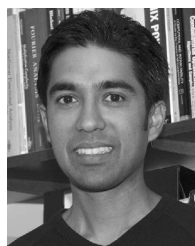
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