STAT 161/261: Homework 1
Bayes Decision Theory. Due April 6 in class.

Prof. Alyson Fletcher

Send all submissions and questions to the TA.

1. Recommended problems: Problem 1.1 and 1.5, Alpayin. **Do not submit**.

2. A person has five coins: three fair coins, one double-headed coin and one double-tailed coin. He selects one of the five coins at random and tosses it.

   (a) What is the probability that the top of the coin that was tossed is a head?

   (b) What is the probability that the bottom of the coin is a head, given that the top of the coin is a head?

3. Suppose that we are trying to estimate a class label $y = 0$ or 1 from a scalar random variable $x \in [-1, 1]$ with likelihood functions

$$p(x|y=0) = \frac{1}{2}, \quad p(x|y=1) = \frac{3}{2}x^2. \tag{1}$$

   (a) Find the maximum likelihood (ML) estimate, $\widehat{y}$, in terms of x.

   (b) For the ML estimate, compute what are called the probabilities of misdetection (P_{MD}) and false alarm (P_{FA}) given by:

$$P_{MD} = \Pr(\widehat{y} = 0|y = 1), \quad P_{FA} = \Pr(\widehat{y} = 1|y = 0). \tag{2}$$

   (c) Let $L(\widehat{y}, y)$ be the loss function

$$L(\widehat{y}, y) = \begin{cases} c_1 & \text{if } \widehat{y} = 1, y = 0, \\ c_2 & \text{if } \widehat{y} = 0, y = 1, \\ 0 & \text{if } \widehat{y} = y, \end{cases} \tag{3}$$

   where c_1 and c_2 are the costs of false alarms and misdetections, respectively. Suppose that

$$P(y = 1) = q, \quad P(y = 0) = 1 - q,$$

   for some q. For the ML detector in part (a), what is the expected Bayes risk $\mathbb{E}(L(\widehat{y}, y))$.

4. Repeat the previous problem with Gaussian likelihoods,

$$p(x|y = i) = \mathcal{N}(x|\mu_i, \sigma_i^2),$$

so that x has mean μ_i and variance σ_i^2 when $y = i$. Assume that $\sigma_1 > \sigma_0$ and $\mu_1 > \mu_0$. Alternatively, just present the solution for one case.

In part (a), the classifier boundaries will be the roots of a quadratic. Do not bother working out those roots. But, do discuss what happens when there are zero, one and two roots. For parts (b) and (c), you will need to express P_D and P_{FA} in terms of the complementary CDF of a Gaussian,

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2} du.$$

Again, leave your answer in terms of the roots of the quadratic.

5. Use MATLAB or equivalent. Suppose that given a binary class label $y = 0$ or 1, we observe a scalar Gaussian random variable,

$$p(x|y = i) = \mathcal{N}(x|\mu_i, \sigma^2),$$

with variance σ^2 and mean μ_0 or μ_1. The class has an a priori probability

$$P(y = 1) = 1 - P(y = 0) = q.$$

Use the parameters $q = 0.3$, $\sigma = 0.25$, $\mu_0 = 0$, $\mu_1 = 1$.

   (a) Find the ML classifier for this problem.

   (b) Generate $N = 10^4$ samples of the data (x_n, y_n) following the distribution of (x, y) above.

   (c) Plot the empirical conditional cumulative distribution functions (CDF) of the samples x_n when $y_n = 0$ or 1. Your plot should plot both CDFs on one plot.

   (d) Run the classifier from part (a) and measure the number of misdetections and false alarms. Compare with the expected values.

6. Problem 2.7 from Alpaydin, 3rd edition.