Homework 5 – Solutions

Ex 4 / page 228

$$f(x) = \begin{cases} \frac{10}{x^2}, & x > 10 \\ 0, & x \leq 10 \end{cases}$$

a) $P(X>20) = \int_{20}^{\infty} \frac{10}{x^2}\,dx = -\frac{10}{x}\Big|_{20}^{\infty} = \frac{10}{20} = \frac{1}{2}.$

b) the cdf: $F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{10}^{x} \frac{10}{u^2}\,du = -\frac{10}{u}\Big|_{10}^{x} = 1 - \frac{10}{x}$, for $x \geq 10$

$$F(x) = \begin{cases} 0, & x < 10 \end{cases}$$

c) $p = P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2}\,dx = -\frac{10}{x}\Big|_{15}^{\infty} = +\frac{10}{15} = \frac{2}{3}$

We then think of a $Y \sim Bi(n,p)$ where $n=6$, $p=\frac{2}{3}$ and we have

$$P(Y \geq 3) = \sum_{i=3}^{6} \binom{6}{i}\left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{6-i} = 1 - \sum_{i=0}^{2} \binom{6}{i}\left(\frac{2}{3}\right)^i\left(\frac{1}{3}\right)^{6-i} = 1 - \left(\left(\frac{1}{3}\right)^6 + 6\cdot\frac{2}{3}\cdot\frac{1}{3^5} + 15\left(\frac{2}{3}\right)^2\cdot\frac{1}{3^4}\right) = 1 - \frac{1+12+60}{3^6} = 1 - \frac{73}{729} = .90$$

Ex 8 / page 229

$$f(x) = x\cdot e^{-x},\ x \geq 0 \Rightarrow EX = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^{\infty} x^2 e^{-x}\,dx =$$

$$= -x^2 e^{-x}\Big|_0^{\infty} + \int_0^{\infty} 2x\,e^{-x}\,dx = -\lim_{x\to\infty}\frac{x^2}{e^x} + 2\underbrace{\int_0^{\infty} f(x)\,dx}_{1 \text{ (density)}} =$$

$$\overset{\text{l'Hospital}}{=} -\lim_{x\to\infty}\frac{2\cdot x}{e^x} + 2 \overset{\text{l'Hospital}}{=} -\lim_{x\to\infty}\frac{2}{e^x} + 2 = \frac{2}{\infty} + 2 = 2.$$

or $EX = \int_0^{\infty} x^2 e^{-x}\,dx = \int_0^{\infty} x^{3-1} e^{-x}\,dx \overset{\text{definition}}{=} \Gamma(3) = 2! = 2.$

Ex 13 / page 229

X = arrival time in minutes , $X \sim U(0,30)$

a) $P(X \geq 10) = \int_{10}^{\infty} f(x)\,dx = \int_{10}^{30} \frac{1}{30}\,dx = \frac{1}{30} x \Big|_{10}^{30} = \frac{20}{30} = \frac{2}{3}$

b) $P(X \geq 25 \mid X \geq 15) = \frac{P(X \geq 25, X \geq 15)}{P(X \geq 15)} = \frac{P(X \geq 25)}{P(X \geq 15)} = \frac{\int_{25}^{30} \frac{1}{30}dx}{\int_{15}^{30} \frac{1}{30}dx} =$

$= \frac{5/30}{15/30} = \frac{1}{3}$.

Ex 21 / page 230

$X \sim N(71, 6.25)$ inches

a) 6 feet 2 inches = 74 inches

$P(X \geq 74) = P\left(\frac{X-71}{\sqrt{6.25}} \geqslant \frac{74-71}{\sqrt{6.25}}\right) = P(Z \geq 1.2) = 1 - \phi(1.2) =$

$\overset{\text{tabel}}{=} 1 - 0.885 = .115.$

b) $P(X \geq 77 \mid X \geq 72) = \frac{P(X \geq 77, X \geqslant 72)}{P(X \geqslant 72)} = \frac{P(X \geqslant 77)}{P(X \geqslant 72)} =$

$P(X \geqslant 77) = P\left(Z \geqslant \frac{77-71}{\sqrt{6.25}}\right) = P(Z \geq 2.4) = 1 - \phi(2.4) = 1 - .979 = .021$

$P(X \geqslant 72) = P\left(Z \geqslant \frac{72-71}{\sqrt{6.25}}\right) = P(Z \geqslant .4) = 1 - \phi(.4) = 1 - .516 = .484$

So $P(X \geqslant 77 \mid X \geqslant 72) = \frac{.021}{.484} = .043 \sim 4.3\%$

Ex 32 / page 231

$X \sim \exp(\frac{1}{2})$

a) $P(X \geq 2) = \int_2^\infty \frac{1}{2} e^{-\frac{x}{2}} dx = -e^{-\frac{x}{2}}\Big|_2^\infty = e^{-\frac{2}{2}} = e^{-1} = .368$

b) $P(X \geq 10 | X \geq 9) = \frac{P(X \geq 10, X \geq 9)}{P(X \geq 9)} = \frac{P(X \geq 10)}{P(X \geq 9)} = \frac{\int_{10}^\infty \frac{1}{2} e^{-\frac{x}{2}} dx}{\int_9^\infty \frac{1}{2} e^{-\frac{x}{2}} dx} =$

$= \frac{e^{-\frac{10}{2}}}{e^{-\frac{9}{2}}} = e^{-\frac{10}{2} + \frac{9}{2}} = e^{-\frac{1}{2}} = .606$

Ex 37 / page 231

$X \sim U(-1,1)$

(a) $P(|X| > \frac{1}{2}) = 1 - P(|X| \leq \frac{1}{2}) = 1 - P(-\frac{1}{2} \leq X \leq \frac{1}{2}) = 1 - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} dx =$

$= 1 - \frac{1}{2} x \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = 1 - \frac{1}{2}(\frac{1}{2} + \frac{1}{2}) = \frac{1}{2} = .5$

(b) $F(x) = P(|X| \leq x) = P(-x \leq X \leq x) = \int_{-x}^{x} \frac{1}{2} du = \frac{1}{2} u \Big|_{-x}^{x} = x$ if $x \in (0,1)$

if $x \geq 1 \Rightarrow \int_{-x}^{x} f(u) du = \int_{-1}^{1} f(u) du = 1 \Rightarrow F(x) = 1$

if $x < 0 \Rightarrow |X| > x \Rightarrow F(x) = 0$. So $F(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x \geq 1 \end{cases}$

Hence $f(x) = F'(x) = \begin{cases} 0, & x \notin [0,1] \\ 1, & x \in [0,1] \end{cases} = 1_{[0,1]}(x)$.

So $|X| \sim U[0,1]$.

[redacted]

Ex 14 / page 233

$X \sim \exp(\lambda)$, $c > 0$ $\Rightarrow$ $cX \sim \exp\left(\frac{\lambda}{c}\right)$

Proof: We shall show that the density of $c \cdot X$ is $\frac{\lambda}{c} \cdot \exp\left(-\frac{\lambda}{c}x\right)$

$$\underset{\substack{\uparrow \\ \text{the cdf of } c\cdot X}}{F_{cX}}(x) = P(c \cdot X \le x) = P\left(X \le \frac{x}{c}\right) = \underset{\substack{\uparrow \\ \text{the cdf of } X}}{F_X}\left(\frac{x}{c}\right)$$

So $$f_{c \cdot X}(x) = F'_{c \cdot X}(x) = F'_X\left(\frac{x}{c}\right) = f_X\left(\frac{x}{c}\right) \cdot \frac{1}{c} = \lambda \cdot e^{-\lambda \cdot \frac{x}{c}} \cdot \frac{1}{c} =$$

$= \frac{\lambda}{c} \cdot e^{-\frac{\lambda}{c} \cdot x}$ which is the density of an $\exp\left(\frac{\lambda}{c}\right)$ ok.