[redacted]

Ex. 1 / page 290

a) Let $A =$ the value on the 1st die ; $B =$ the value on the 2nd die

Then $X = \max(A, B)$, $Y = A+B$; $X \in \{1,2,\dots,6\}$, $Y \in \{2,3,\dots,12\}$

We need to find $P(X=i, Y=j)$, for $i \in \{1,\dots,6\}$, $j \in \{2,3,\dots,12\}$

First let us notice that $\underset{Y}{\underset{\shortparallel}{A+B}} \geq \underset{X}{\underset{\shortparallel}{\max(A,B)}}$ for any $A, B \geq 1$.

So $P(X=i, Y=j) = 0$ if $j \leq i$. (**)

Now take $j > i$.

$P(X=i, Y=j) = P(\max(A,B)=i, A+B=j) = P(\max(A,B)=i, A+B=j, A>B) +$

$+ P(\max(A,B)=i, A+B=j, A\leq B) = P(A=i, A+B=j, A>B) + P(B=i, A+B=j, A\leq B)$

$= P(A=i, B=j-i, A>B) + P(B=i, A=j-i, A\leq B)$. (*)

(1) But $P(A=i, B=j-i, A>B) = 0$ if $i \leq j-i \Leftrightarrow j \geq 2i$

and $= P(A=i, B=j-i) = P(A=i)\cdot P(B=j-i) = \frac{1}{36}$, if $i > j-i \Leftrightarrow j < 2i$

(2) and $P(B=i, A=j-i, A\leq B) = 0$ if $j-i > i \Leftrightarrow j > 2i$.

$= P(B=i)P(A=j-i) = \frac{1}{36}$, if $j-i \leq i \Leftrightarrow j \leq 2i$

From (1) & (2) and using (*), (**) we get

$$P(X=i, Y=j) = \begin{cases} 0 & , \text{if } j \leq i \\ \frac{2}{36} & , \text{if } i<j<2i \\ \frac{1}{36} & , \text{if } j = 2i \\ 0 & , \text{if } j > 2i . \end{cases}$$

b) $X = A$

$Y = \max(A,B)$

So $Y \geq X$. Therefore $P(X=i, Y=j)=0$ if $i>j$.

Take $i \leq j$:

$P(X=i, Y=j) = P(A=i, \max(A,B)=j) = P(A=i, \max(A,B)=j, A \geq B) +$

$+ P(A=i, \max(A,B)=j, A<B) = P(A=i, A=j, A\geq B) + P(A=i, B=j, A<B)$ (*)

If $i=j$ then $P(X=i, Y=i) = P(A=i, B\leq i) + \underbrace{P(A=i, B=i, A<B)}_{0} =$

$= P(A=i)P(B\leq i) = \frac{1}{6}\cdot(P(B=1)+P(B=2)+\dots+P(B=i)) = \frac{1}{6}(\underbrace{\tfrac{1}{6}+\dots+\tfrac{1}{6}}_{i}) = \frac{i}{36}$

If $i<j$ then by (*) $P(X=i, Y=j) = \underbrace{P(A=i, A=j, A\geq B)}_{0} + P(A=i, B=j, \underset{\text{satisfied}}{A<B}) =$

$= P(A=i, B=j) = \frac{1}{36}$.

So $P(X=i, Y=j) = \begin{cases} 0, & \text{if } i>j \\ \frac{i}{36}, & \text{if } i=j \\ \frac{1}{36}, & \text{if } i<j. \end{cases}$

c) $X=\min(A,B)$, $Y=\max(A,B)$. Again $Y\geq X$ so $P(X=i, Y=j)=0$ if $j<i$.

Let $j=i$: $P(X=i, Y=i) = P(\min(A,B)=i, \max(A,B)=i) = P(A=i, B=i) = \frac{1}{36}$.

Let $i<j$: $P(X=i, Y=j) = P(\min(A,B)=i, \max(A,B)=j) = P(\min(A,B)=i, \max(A,B)=j, A\geq B) +$

$+ P(\min(A,B)=i, \max(A,B)=j, A<B) = P(A=j, B=i) + P(A=i, B=j) = \frac{2}{36}$.

So $P(X=i, Y=j) = \begin{cases} 0, & \text{if } i>j \\ \frac{1}{36}, & \text{if } i=j \\ \frac{2}{36}, & \text{if } i<j \end{cases}$.

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$$f(x,y)=c(y^2-x^2)e^{-y} \quad , -y\le x\le y \;,\; y>0.$$

a) f is a density so $\iint f(x,y)\,dx\,dy=1$

But $\iint f(x,y)\,dx\,dy = \int_0^\infty \int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx\,dy = \int_0^\infty c\cdot e^{-y}\int_{-y}^{y}(y^2-x^2)\,dx\,dy$

$$= \int_0^\infty c e^{-y}\left[y^2\cdot x\Big|_{-y}^{y} - \frac{x^3}{3}\Big|_{-y}^{y}\right]dy = \int_0^\infty c\cdot e^{-y}\cdot\left(2y^3-\frac{2}{3}y^3\right)dy =$$

$$= c\int_0^\infty e^{-y}\cdot\frac{4}{3}y^3\,dy = \frac{4}{3}c\int_0^\infty e^{-y}\cdot y^3\,dy = \frac{4}{3}c\cdot\Gamma(4)=\frac{4}{3}c\cdot 3! = 8c$$

So $8c=1$ and therefore $c=\frac{1}{8}$

b) $EX = \int x\, f_X(x)\,dx.$

$$f_X(x) = \int_{|x|}^{\infty} f(x,y)\,dy = \int_{|x|}^{\infty}\frac{1}{8}\cdot(y^2-x^2)e^{-y}\,dy = \frac{1}{8}\int_{|x|}^{\infty} y^2e^{-y}\,dy - \frac{x^2}{8}\int_{|x|}^{\infty} e^{-y}\,dy$$

$$\int_{|x|}^{\infty} y^2e^{-y}\,dy = -y^2e^{-y}\Big|_{|x|}^{\infty} + \int_{|x|}^{\infty}2ye^{-y}\,dy = x^2\cdot e^{-|x|} + 2ye^{-y}\Big|_{|x|}^{\infty} + 2\int_{|x|}^{\infty}e^{-y}\,dy =$$

$$= x^2\cdot e^{-|x|} + 2|x|e^{-|x|} + 2\cdot e^{-|x|} = (x^2+2|x|+2)e^{-|x|}.$$

$$\int_{|x|}^{\infty} e^{-y}\,dy = e^{-|x|}$$

So $f_X(x) = \frac{1}{8}(x^2+2|x|+2)e^{-|x|} - \frac{x^2}{8}\cdot e^{-|x|} = \frac{1}{4}(|x|+1)e^{-|x|}.$

$$\Rightarrow EX = \int_{-\infty}^{+\infty} x\cdot\frac{1}{4}(|x|+1)e^{-|x|}\,dx = \int_{-\infty}^{0} x\cdot\frac{1}{4}(|x|+1)e^{-|x|}\,dx + \int_{0}^{\infty} x\cdot\frac{1}{4}(|x|+1)e^{-|x|}\,dx$$

$$= -\int_0^\infty x\cdot\frac{1}{4}(|x|+1)e^{-|x|}\,dx + \int_0^\infty x\cdot\frac{1}{4}(|x|+1)e^{-|x|}\,dx = 0.$$

$$f_Y(y) = \int_{-y}^{y} f(x,y)\,dx = \int_{-y}^{y}\frac{1}{8}(y^2-x^2)\cdot e^{-y}\,dx = \frac{1}{8}e^{-y}\int_{-y}^{y}(y^2-x^2)\,dx =$$

$$= \frac{1}{8}e^{-y}\left(y^2x-\frac{x^3}{3}\right)\Big|_{-y}^{y} = \frac{1}{8}e^{-y}\cdot\left(2y^3-\frac{2}{3}y^3\right) = \frac{4}{3}\cdot\frac{1}{8}e^{-y}\cdot y^3 = \frac{1}{6}y^3e^{-y},\ y>0.$$

Ex 10 / page 291

$f(x,y) = e^{-(x+y)}$, $x,y \ge 0$.

a) $P(X<Y) = \iint_D f(x,y)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2_+ \mid x<y\}$

So $P(X<Y) = \int_0^\infty \int_x^\infty f(x,y)\,dy\,dx = \int_0^\infty \int_x^\infty e^{-(x+y)}\,dy\,dx =$

$= \int_0^\infty e^{-x}\cdot(-e^{-y})\Big|_x^\infty dx = \int_0^\infty e^{-x}\cdot e^{-x}\,dx = \int_0^\infty e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_0^\infty = \frac{1}{2}$.

b) $P(X<a) = \int_0^a f_X(x)\,dx$

$f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty e^{-x-y}\,dy = -e^{-x-y}\Big|_0^\infty = e^{-x}$.

So $P(X<a) = \int_0^a e^{-x}\,dx = -e^{-x}\Big|_0^a = 1-e^{-a}$.

Theoretical #5 : a) $Z = X/Y$

For $a>0$, $F_Z(a) = P(X \le aY) = \int_0^\infty \int_0^{ay} f_X(x) f_Y(y)\,dx\,dy =$

$= \int_0^\infty f_Y(y) \int_0^{ay} f_X(x)\,dx\,dy = \int_0^\infty f_Y(y) F_X(ay)\,dy$

$\int_0^\infty f_Y(y)\cdot y \cdot f_X(ay)\,dy$

$f_Z(a) = \frac{\partial}{\partial a}F_Z(a) = \frac{\partial}{\partial a}\int_0^\infty f_Y(y)F_X(ay)\,dy = \int_0^\infty f_Y(y)\frac{\partial}{\partial a}F_X(ay)\,dy = \cancel{\int_0^\infty f_Y(y)\cdot y\cdot f_X(ay)\,dy}$

If $X \sim exp(\lambda)$, $Y \sim exp(\mu)$ then $f_Z(a) = \int_0^\infty \mu e^{-\mu y}\, y\, \lambda \cdot e^{-\lambda a y}\,dy =$

$= \int_0^\infty \mu\lambda\, y \cdot e^{-(\mu+\lambda a)y}\,dy = \underbrace{-\frac{\mu\lambda}{\mu+\lambda a}\cdot e^{-(\mu+\lambda a)y}\cdot y\Big|_0^\infty}_{0} + \frac{\mu\lambda}{\mu+\lambda a}\cdot\int_0^\infty e^{-(\mu+\lambda a)y}\,dy$

$= \frac{\mu\lambda}{\mu+\lambda a}\cdot\left(-\frac{e^{-(\mu+\lambda a)y}}{\mu+\lambda a}\right)\Big|_0^\infty = \frac{\mu\lambda}{(\mu+\lambda a)^2}$.

b) $F_Z(a) = P(XY \le a) = \int_0^\infty \int_0^{a/y} f_X(x) f_Y(y)\,dx\,dy = \int_0^\infty F_X(a/y) f_Y(y)\,dy \Rightarrow f_Z(a) = \int_0^\infty f_X(a/y)\frac{1}{y} f_Y(y)\,dy$

So if $X \sim exp(\lambda)$, $Y \sim exp(\mu)$ then $f_Z(a) = \int_0^\infty \lambda\mu \cdot \frac{1}{y}\cdot e^{-\frac{\lambda a}{y}}\cdot e^{-\mu y}\,dy$

Ex 3/page 380

$X, Y \sim U(0,1)$

$$E[|X-Y|^a] = \int_0^1 \int_0^1 |x-y|^a \, dy\, dx =$$

$$\int_0^1 |x-y|^a dy = \int_0^x (x-y)^a dy + \int_x^1 (y-x)^a dy = \int_0^x u^a du + \int_0^{1-x} u^a du = \frac{x^{a+1} + (1-x)^{a+1}}{a+1}$$

Hence, $E[|X-Y|^a] = \frac{1}{a+1} \cdot \int_0^1 [x^{a+1} + (1-x)^{a+1}] dx = \frac{2}{(a+1)(a+2)}$

Ex 5/page 380 The joint density of the point (X,Y) at which the accident occurs is $f(x,y) = \frac{1}{9}$, $-\frac{3}{2} < x, y < \frac{3}{2}$

$$= f(x) f(y) \quad \text{where} \quad f(a) = \frac{1}{3}, \ -\frac{3}{2} < a < \frac{3}{2}$$

Hence we may conclude that X and Y are independent and uniformly distributed on $(-\frac{3}{2}, \frac{3}{2})$. Therefore,

$$E[|X| + |Y|] = \int |x| f_X(x) dx + \int |y| f_Y(y) dy = 2 \int_{-\frac{3}{2}}^{\frac{3}{2}} |x| \cdot \frac{1}{3} dx =$$

$$= 4 \int_0^{\frac{3}{2}} |x| \cdot \frac{1}{3} dx = 4 \int_0^{\frac{3}{2}} x \cdot \frac{1}{3} dx = \frac{4}{3} \cdot \frac{x^2}{2} \Big|_0^{\frac{3}{2}} = \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}$$

Theoretical: #4 /page 390

$EX = \mu$, $\text{Var } X = \sigma^2$: We can develop $g(x)$ using Taylor around μ.

~~$E[g(X)]$~~ So $g(x) \approx g(\mu) + (x-\mu) g'(\mu) + \frac{(x-\mu)^2}{2!} g''(\mu) \Longrightarrow$

$$\Rightarrow Eg(X) \approx E\left(g(\mu) + (X-\mu) g'(\mu) + \frac{(X-\mu)^2}{2!} g''(\mu)\right) = g(\mu) + E(X-\mu) g'(\mu) + \frac{1}{2} E(X-\mu)^2 g''(\mu)$$

$$= g(\mu) + \underbrace{(EX-\mu)}_{0} \cdot g'(\mu) + \frac{1}{2} \text{Var } X \cdot g''(\mu) = g(\mu) + \frac{1}{2} \sigma^2 \cdot g''(\mu) \quad \text{ok.}$$