UCLA STAT 100A, Final Exam Review Guide

Chapter 4: Random Variables

4.1 Random Variables

A random variable is a "rule" (or, more technically, a function) which assigns a number to each outcome in the sample space of an experiment. Probabilities are then assigned to the values of the random variable.

Exercise 4.1 (Random Variables)

1. *Flipping A Coin Twice.* Let random variable X be the number of heads that come up.

   (a) $P\{HH\} =$ (circle one) $\boldsymbol{P\{X=0\}}$ / $\boldsymbol{P\{X=1\}}$ / $\boldsymbol{P\{X=2\}}$

   (b) $P\{HT, HT\} =$ (circle one) $\boldsymbol{P\{X=0\}}$ / $\boldsymbol{P\{X=1\}}$ / $\boldsymbol{P\{X=2\}}$

   (c) $P\{TT\} =$ (circle one) $\boldsymbol{P\{X=0\}}$ / $\boldsymbol{P\{X=1\}}$ / $\boldsymbol{P\{X=2\}}$

   (d) **True** / **False** If the coin is fair (heads come up as often as tails), the distribution of X (the number of heads in two flips of the coin) is then

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{2}$

   (e) **True** / **False** If heads come up twice as often as tails, the distribution of X is then

x	0	1	2
$P(X=x)$	$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$	$2 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$	$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

2. *Rolling a Pair of Dice.*

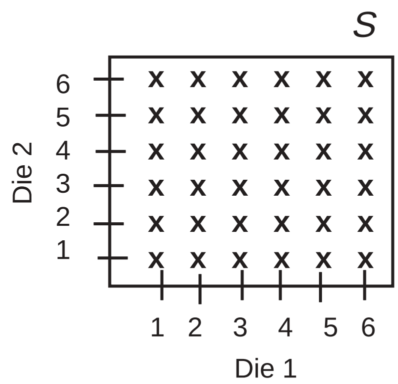


Figure 4.1 (Sample Space For Rolling A Pair of Dice)

(a) Let X be the sum of the dice.
$P\{(1,1)\}$ = (circle one) $\boldsymbol{P\{X=1\}}$ / $\boldsymbol{P\{X=2\}}$ / $\boldsymbol{P\{X=3\}}$
$P\{(1,2),(2,1)\}$ = (circle one) $\boldsymbol{P\{X=1\}}$ / $\boldsymbol{P\{X=2\}}$ / $\boldsymbol{P\{X=3\}}$
$P\{(1,5),(2,4),(3,3),(2,4),(1,5)\}$ =
(circle one) $\boldsymbol{P\{X=4\}}$ / $\boldsymbol{P\{X=5\}}$ / $\boldsymbol{P\{X=6\}}$
$P\{X=11\}$ = (circle *best* one) $\boldsymbol{P\{(5,6)\}}$ / $\boldsymbol{P\{(6,5)\}}$ / $\boldsymbol{P\{(5,6),(6,5)\}}$
True / **False** If the dice are fair (each number comes up one sixth of the time), the distribution of X (the sum of two rolls of a pair of dice) is then

x	2	3	4	5	6	7
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$

x	8	9	10	11	12
$P(X=x)$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(b) Let X be the number of 4's rolled.
$P\{(4,4)\}$ = (circle one) $\boldsymbol{P\{X=0\}}$ / $\boldsymbol{P\{X=1\}}$ / $\boldsymbol{P\{X=2\}}$
$P\{X=1\}$ = (circle *best* one)
$\boldsymbol{P\{(1,4),(2,4),(3,4),(5,4),(6,4)\}}$
$\boldsymbol{P\{(4,1),(4,2),(4,3),(4,5),(4,6)\}}$
$\boldsymbol{P\{(1,4),(2,4),(3,4),(5,4),(6,4),(4,1),(4,2),(4,3),(4,5),(4,6)\}}$
$P\{X=0\}$ = (circle one) $\boldsymbol{\frac{11}{36}}$ / $\boldsymbol{\frac{20}{36}}$ / $\boldsymbol{\frac{25}{36}}$
True / **False** If the dice are fair, the distribution of X (the number of 4's in two rolls of a pair of dice) is then

x	0	1	2
$P(X=x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

3. *Flipping Until a Head Comes Up.* A (weighted) coin has a probability of $p=0.7$ of coming up heads (and so a probability of $1-p=0.3$ of coming up tails). This coin is flipped until a head comes up or until a total of 4 flips are made. Let X be the number of flips.

(a) $P\{X = 1\} = P\{H\}$ (circle one) **0.7** / **0.3** / **0.3(0.7)**

(b) $P\{X = 2\} = P\{TH\}$ (circle one) **0.7** / **0.3** / **0.3(0.7)**

(c) $P\{X = 3\} = P\{TTH\}$ (circle one) **0.7** / **0.3(0.7)** / $\mathbf{0.3^2(0.7)}$

(d) $P\{X = 4\} = P\{TTTT, TTTH\}$ (circle none, one or more)
$\mathbf{0.7}$ / $\mathbf{0.3^3(0.3 + 0.7)}$ / $\mathbf{0.3^3}$
(Remember that, at most, only four (4) flips can be made.)

4. *Roulette.* The roulette table has 38 numbers: the numbers are 1 to 36, 0 and 00. A ball is spun on a corresponding roulette wheel which, after a time, settles down and the ball drops into one of 38 slots which correspond to the 38 numbers on the roulette table. Consider the following roulette table.

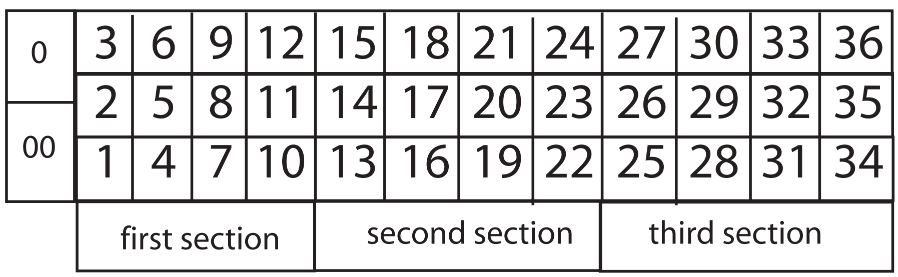


Figure 4.2 (Roulette)

(a) The sample space consists of 38 outcomes: {00, 0, 1, . . . , 35, 36}. The event "an even comes up" (numbers 2, 4, 6, . . . , 36, but not 0 or 00) consists of (circle one) **18** / **20** / **22** numbers.
The chance an even comes up is then (circle one) **18/38** / **20/38** / **22/38**
The event "a number in the second section comes up" (12 numbers: 13, 16, 19, 14, 17, 20, 15, 18 and 21) consists of
(circle one) **12** / **20** / **22** numbers.
The chance a second section comes up is then
(circle one) $\mathbf{\frac{12}{38}}$ / $\mathbf{\frac{20}{38}}$ / $\mathbf{\frac{22}{38}}$

(b) Let random variable X be the winnings from a \$1 bet placed on an even coming up. If an even number does come up, the gambler keeps his dollar and receives another dollar (+\$1). If an odd number comes up, the gambler loses the dollar he bets (−\$1). In other words, an even pays "1 to 1". And so
$P\{X = \$1\} =$ (circle one) $\mathbf{\frac{18}{38}}$ / $\mathbf{\frac{20}{38}}$ / $\mathbf{\frac{22}{38}}$
$P\{X = -\$1\} =$ (circle one) $\mathbf{\frac{18}{38}}$ / $\mathbf{\frac{20}{38}}$ / $\mathbf{\frac{22}{38}}$
True / **False** The distribution of X is then

x	-\$1	\$1
$P(X = x)$	$\frac{22}{38}$	$\frac{20}{38}$

(c) Let random variable Y be the winnings from a \$1 bet placed on the second section coming up. If a second section number does come up, the gambler keeps his dollar and receives another two dollars (+\$2). If a first or third section number comes up, the gambler loses the dollar he bets (−\$1). In other words, an second section bet pays "2 to 1".
$P\{Y = \$2\}$ = (circle one) $\mathbf{\frac{12}{38}}$ / $\mathbf{\frac{20}{38}}$ / $\mathbf{\frac{26}{38}}$
$P\{Y = -\$1\}$ = (circle one) $\mathbf{\frac{12}{38}}$ / $\mathbf{\frac{20}{38}}$ / $\mathbf{\frac{26}{38}}$
True / **False** The distribution of Y is then

y	-\$1	\$2
$P(Y = y)$	$\frac{26}{38}$	$\frac{12}{38}$

By the way, $P\{Y = \$1\}$ = (circle one) **0** / $\mathbf{\frac{20}{38}}$ / $\mathbf{\frac{26}{38}}$

5. *Random Variables And Urns.* Two marbles are taken, one at a time, with*out* replacement, from an urn which has 6 red and 10 blue marbles. We win \$2 for each red marble chosen and lose \$1 for each blue marble chosen. Let X be the winnings.

(a) The chance both marbles are red is

(circle one) $\dfrac{\binom{6}{2}\binom{10}{0}}{\binom{16}{2}}$ / $\dfrac{\binom{9}{1}\binom{8}{2}}{\binom{16}{3}}$ / $\dfrac{\binom{8}{1}\binom{11}{2}}{\binom{16}{3}}$

(b) Since the winnings are $X = \$4$, if both marbles are red, then
$P\{X = \$4\}$ = (circle one) **0.025** / **0.125** / **0.225**
Use your calculator to work out the combinations.

(c) Choose the correct distribution below.

i. Distribution A.

x	-\$2	\$1	\$4
$P(X = x)$	0.500	0.375	0.125

ii. Distribution B.

x	-\$2	\$1	\$4
$P(X = x)$	0.375	0.500	0.125

4.2 Distribution Functions

The (cumulative) distribution function (c.d.f.) is

$$F(b) = P\{X \leq b\}$$

where $-\infty < b < \infty$.

Exercise 4.2 (Cumulative Distribution Function)

1. *Flipping A Coin Twice.* Recall,

x	0	1	2
$P\{X = x\}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{2}$

or

$$P\{X = 0\} = 0.25, \ P\{X = 1\} = 0.50, \ P\{X = 2\} = 0.25.$$

Consequently,

(a) $P\{X \leq 0\} =$ (circle one) **0.25** / **0.75** / **1**

(b) $P\{X \leq 1\} =$ (circle one) **0.25** / **0.75** / **1**

(c) $P\{X \leq 2\} =$ (circle one) **0.25** / **0.75** / **1**

(d) **True** / **False** Since $F(b) = P\{X \leq b\}$,

$$F(0) = 0.25, \ F(1) = 0.75, \ F(2) = 1.$$

or

$$F(x) = \begin{cases} 0.25, & x < 0 \\ 0.75, & 0 \leq x < 1 \\ 1, & 1 \leq x \end{cases}$$

(e) **True** / **False** A graph of the distribution function is given below.

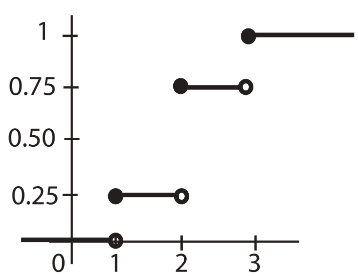


Figure 4.3 (Graph of Distribution Functions)

2. *Rolling a Pair of Dice.*

(a) Let X be the sum of the dice. Recall,

x	2	3	4	5	6	7
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$

x	8	9	10	11	12
$P(X = x)$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Consequently,
$F(2)P\{X \leq 2\} =$ (circle one) $\mathbf{\frac{1}{36}}$ / $\mathbf{\frac{2}{36}}$ / $\mathbf{\frac{3}{36}}$
$F(3) = P\{X \leq 3\} =$ (circle one) $\mathbf{\frac{1}{36}}$ / $\mathbf{\frac{2}{36}}$ / $\mathbf{\frac{3}{36}}$
$F(11) =$ (circle one) $\mathbf{\frac{34}{36}}$ / $\mathbf{\frac{35}{36}}$ / $\mathbf{1}$
$F(12) =$ (circle one) $\mathbf{\frac{34}{36}}$ / $\mathbf{\frac{35}{36}}$ / $\mathbf{1}$
True / **False**

$$F(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{36}, & 2 \leq x < 3 \\ \frac{3}{36}, & 3 \leq x < 4 \\ \vdots & \vdots \\ \frac{35}{36}, & 11 \leq x < 12 \\ 1, & 12 \leq x \end{cases}$$

$P\{X < 2\} =$ (circle one) $\mathbf{\frac{0}{36}}$ / $\mathbf{\frac{1}{36}}$ / $\mathbf{1}$
$P\{X > 2\} =$ (circle one) $\mathbf{\frac{0}{36}}$ / $\mathbf{\frac{35}{36}}$ / $\mathbf{1}$
$P\{2 \leq X < 4\} =$ (circle none, one or more) $\boldsymbol{F(4) - F(2)}$ / $\boldsymbol{F(3) - F(1)}$ / $\boldsymbol{F(3) - F(1)}$

(b) Let X be the number of 4's rolled. Recall,

$$P\{X = 0\} = \frac{25}{36},\ P\{X = 1\} = \frac{10}{36},\ P\{X = 2\} = \frac{1}{36}.$$

Consequently,
$F(0) =$ (circle one) $\mathbf{\frac{25}{36}}$ / $\mathbf{\frac{35}{36}}$ / $\mathbf{1}$
$F(1) =$ (circle one) $\mathbf{\frac{25}{36}}$ / $\mathbf{\frac{35}{36}}$ / $\mathbf{1}$
$F(2) =$ (circle one) $\mathbf{\frac{25}{36}}$ / $\mathbf{\frac{35}{36}}$ / $\mathbf{1}$
True / **False**

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{25}{36}, & 0 \leq x < 1 \\ \frac{35}{36}, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases}$$

3. *Another Distribution.* Let

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{3}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases}$$

(a) $P\{X = 0\} = F(0) =$ (circle one) $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{1}{3}}$ / $\mathbf{\frac{1}{2}}$

(b) $P\{X = 1\} = F(1) - F(0) =$ (circle one) $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{1}{3}}$ / $\mathbf{\frac{1}{2}}$

(c) $P\{X = 2\} = F(2) - F(1) =$ (circle one) $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{1}{3}}$ / $\mathbf{\frac{1}{2}}$

4. *Properties of Distribution Functions.* Circle true or false.

   (a) **True** / **False** If $a < b$, then $F(a) \le F(b)$; that is, F is nondecreasing.

   (b) **True** / **False** $\lim_{b\to\infty} F(b) = 1$

   (c) **True** / **False** $\lim_{b\to-\infty} F(b) = 0$

   (d) **True** / **False** $\lim_{n\to\infty} F(b_n) = F(b)$; that is, F is right continuous (which determines where the solid and empty endpoints are on the graph of a distribution function)

4.3 Discrete Random Variables

The random variable is *discrete* if it assigns the outcomes in a sample space to a set of finite or countably infinite possible real values. We introduce the notation

$$p(a) = P\{X = a\}$$

Exercise 4.3 (Discrete Random Variables)

1. *Chance of Seizures.* The number of seizures, X, of a typical epileptic person in any given year is given by the following probability distribution.

X	0	2	4	6	8	10
p(x)	0.17	0.21	0.18	0.11	0.16	0.17

   (a) The chance a person has 8 epileptic seizures is $p(8) =$(circle one) **0.17** / **0.21** / **0.16** / **0.11**.

   (b) The chance a person has less than 6 seizures is (circle one) **0.17** / **0.21** / **0.56** / **0.67**.

   (c) $P\{X \le 4\} =$ (circle one) **0.17** / **0.21** / **0.56** / **0.67**.

   (d) $p(2) =$ (circle one) **0.17** / **0.21** / **0.56** / **0.67**.

   (e) $p(2.1) =$ (circle one) **0** / **0.21** / **0.56** / **0.67**.

   (f) $P\{X > 2.1\} =$ (circle one) **0.21** / **0.38** / **0.56** / **0.62**.

   (g) $P\{X = 0\} + P\{X = 2\} + P\{X = 4\} + P\{X = 6\} + P\{X = 8\} + P\{X = 10\} =$
   (circle one) **0.97** / **0.98** / **0.99** / **1**.

(h) *Histogram (Graph) of Distribution.* Consider the probability histograms given in the figure below.

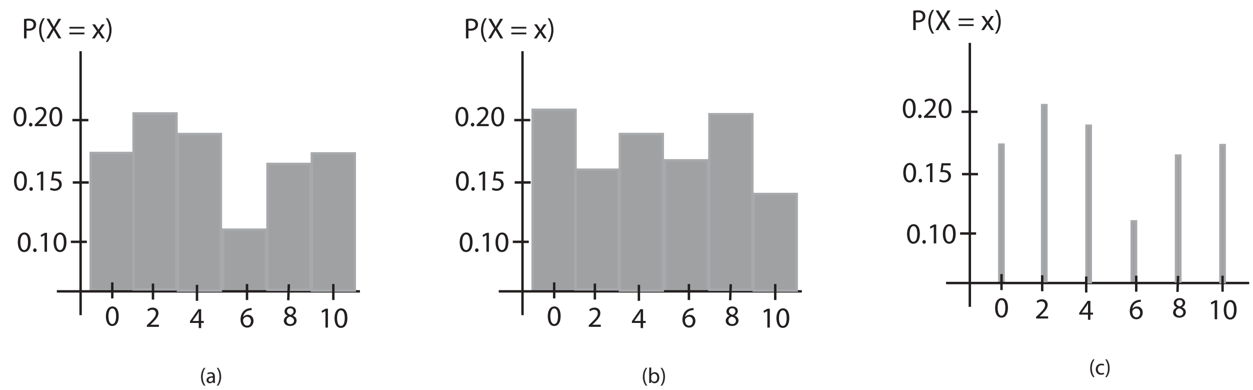


Figure 4.4 (Probability Histogram)

Which, if any, of the three probability histograms in the figure above, describe the probability distribution of the number of seizures?
(circle none, one or more) **(a)** / **(b)** / **(c)**

(i) *Function of Distribution.* Which one of the following functions describes the probability distribution of the number of seizures?

i. Function (a).

$$P(X = x) = \begin{cases} 0.17, & \text{if } x = 0 \\ 0.21, & \text{if } x = 2 \end{cases}$$

ii. Function (b).

$$P(X = x) = \begin{cases} 0.18, & \text{if } x = 4 \\ 0.11, & \text{if } x = 6 \end{cases}$$

iii. Function (c).

$$P(X = x) = \begin{cases} 0.17, & \text{if } x = 0 \\ 0.21, & \text{if } x = 2 \\ 0.18, & \text{if } x = 4 \\ 0.11, & \text{if } x = 6 \\ 0.16, & \text{if } x = 8 \\ 0.17, & \text{if } x = 10 \end{cases}$$

2. *Chance of Being A Smoker.* Consider the following distribution of the number of smokers in a group of three people,

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(a) At *exactly* $x = 0$, $P(X = 0) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

(b) whereas at $x = -2$, $P(X = -2) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

(c) and, indeed, at any $x < 0$, $P(X = x) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

(d) Since, also, at $x = \frac{1}{4}$, $P(X = \frac{1}{4}) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

(e) and at $x = \frac{1}{2}$, $P(X = \frac{1}{2}) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

(f) and, indeed, at any $0 < x < 1$, $P(X = x) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

(g) But, at *exactly* $x = 1$, $P(X = 1) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

(h) whereas, at $x = 1\frac{1}{4}$, $P(X = 1\frac{1}{4}) =$ (circle one) $\mathbf{0}$ / $\mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$

3. *Number of Bikes.* The number of bicycles, X, on a bike rack at lunch time during the summer is given by the following probability distribution.

$$p(x) = \frac{1}{5}, \quad x = 5, 6, 7, 8, 9,$$

(a) The chance the bike rack has 8 bicycles is
$p(8) =$ (circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(b) The chance the bike rack has less than 8 bicycles is
(circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(c) $P\{X \leq 6\} =$ (circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(d) $p(7) =$ (circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(e) $p(8.1) =$ (circle one) $\mathbf{\frac{0}{5}}$ / $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$.

(f) $P\{5 < X < 8\} =$ (circle one) $\mathbf{\frac{0}{5}}$ / $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$.

(g) $\sum_{x=5}^{x=9} p(x) =$ (circle one) $\mathbf{\frac{0}{5}}$ / $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{5}{5}}$.

4. *Flipping a Coin.* The number of heads, X, in one flip of a coin, is given by the following probability distribution.

$$p(x) = (0.25)^x(0.75)^{1-x}, \quad x = 0, 1$$

(a) The chance of flipping 1 head ($X = 1$) is
$p(1) = (0.25)^1(0.75)^{1-1} =$ (circle one) **0** / **0.25** / **0.50** / **0.75**.

(b) This coin is (circle one) **fair** / **unfair**.

(c) The chance of flipping no heads ($X = 0$) is
$p(0) = (0.25)^0(0.75)^{1-0} =$ (circle one) **0** / **0.25** / **0.50** / **0.75**.

(d) A "tabular" version of this probability distribution of flipping a coin is (circle one)

i. Distribution A.

X	0	1
$p(x)$	0.25	0.75

ii. Distribution B.

X	0	1
$p(x)$	0.75	0.25

iii. Distribution C.

X	0	1
$p(x)$	0.50	0.50

(e) The number of different ways of describing a distribution include (check none, one or more)

i. function

ii. tree diagram

iii. table

iv. graph

(f) **True** / **False** $F(a) = \sum_{all x \leq a} p(x)$

5. *Rock, Scissors and Paper. Rock, scissors and paper* (RSP) involves two players in which both can either show either a "rock" (clenched fist), "scissors" (V–sign) or "paper" (open hand) simultaneously, where either does not know what the other is going to show in advance. Rock beats scissors (crushes it), scissors beats paper (cuts it) and paper bets rock (covers it). Whoever wins, receives a dollar ($1). The *payoff* matrix RSP is given below. Each element represents the
amount player C (column) pays player R (row).

Player C → / Player R ↓	rock (1)	scissors (2)	paper (3)
rock (1)	0	$1	-$1
scissors (2)	-$1	0	$1
paper (3)	$1	-$1	0

(a) According to the payoff matrix, if both players C and R show "rock", then player C pays player R
(circle one) -$**1** / $**0** / $**$1**
(In other words, no one wins–one player does not pay the other player.)

(b) If player C shows "rock" and player R shows "paper", then player C pays player R
(circle one) -$**1** / $**0** / $**$1**

(c) To say player C pays player R *negative* one dollar, -$1, means (circle one)

i. player C pays player R one dollar.
ii. player R pays player C one dollar.
iii. player C loses one dollar (to player R).
iv. player R wins one dollar (from player C).

(d) *If* each of the nine possible outcomes are equally likely (each occur with a probability of $\frac{1}{9}$), which is the correct probability distribution of payoff X, the amount that player C pays player R

i. Distribution A.

X	-1	0	1
$p(x)$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$

ii. Distribution B.

X	-1	0	1
$p(x)$	$\frac{3}{9}$	$\frac{3}{9}$	$\frac{3}{9}$

6. *Binomial Distribution.* The distribution of the binomial random variable is given by

$$p(i) = P\{X = i\} = \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i}, \ i = 0, 1, \ldots, n$$

(a) If $n = 10$, $p = 0.65$, $i = 4$, then
$p(5) = \left(\begin{array}{c} 10 \\ 4 \end{array} \right) 0.65^4 0.35^6 =$ (circle one) **0.025** / **0.050** / **0.069**
(2nd DISTR 0:binompdf(10,0.65,4) ENTER)

(b) If $n = 11$, $p = 0.25$, $i = 3$, then
$p(3) =$ (circle one) **0.26** / **0.50** / **0.69**

(c) **True** / **False** If $n = 4$, $p = 0.25$, then the entire distribution is given by

X	0	1	2	3	4
$p(x)$	0.32	0.42	0.21	0.05	0.004

(2nd DISTR 0:binompdf(4,0.25) ENTER)

7. *Poisson Distribution.* The distribution of the Poisson random variable is given by

$$p(i) = P\{X = i\} = e^{-\lambda} \frac{\lambda^i}{i!}, \ i = 0, 1, \ldots, \lambda > 0$$

(a) If $\lambda = 10$, $i = 4$, then
$p(4) = P\{X = 4\} = e^{-10} \frac{10^4}{4!} =$ (circle one) **0.019** / **0.050** / **0.069**
(2nd DISTR B:poissonpdf(10,4) ENTER)

(b) If $n = 11$, $i = 3$, then
$p(3) =$ (circle one) **0.0026** / **0.0037** / **0.0069**

4.4 Expected Value

The *expected value*, $E[X]$ (or *mean*, μ), of a random variable, X is given by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

It is, roughly, a weighted average of the probability distribution.

Exercise 4.4 (Expected Value of a Discrete Random Variable)

1. *Seizures.* The probability mass function for the number of seizures, X, of a typical epileptic person in any given year is given in the following table.

X	0	2	4	6	8	10
$p(x)$	0.17	0.21	0.18	0.11	0.16	0.17

(a) *A First Look: Expected Value Is Like The Fulcrum Point of Balance.*

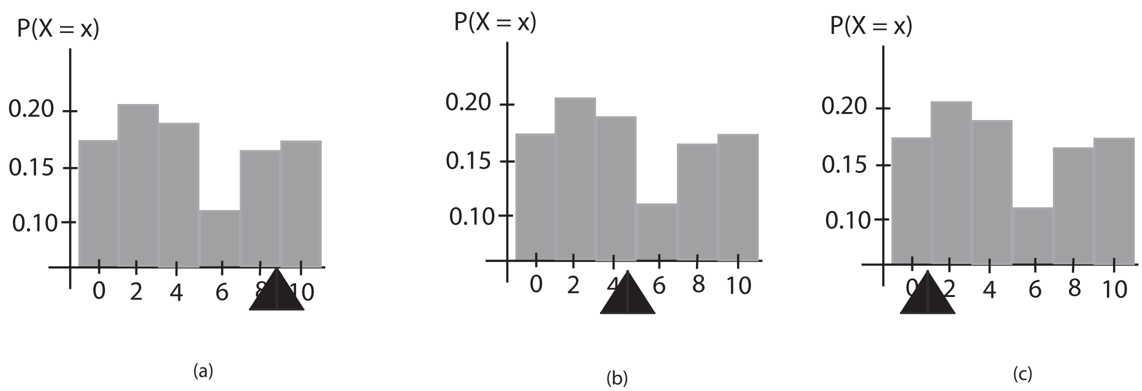


Figure 4.5 (Expected Value Is Like The Fulcrum Point of Balance)

If the expected value is like a fulcrum point which *balances* the "weight" of the probability distribution, then the expected value is most likely close to the point of the fulcrum given in which of the three graphs above? Circle one. **(a)** / **(b)** / **(c)**
In other words, the expected value seems close to (circle one) **1** / **5** / **9**

(b) *Calculating The Expected Value.* The expected value (mean) number of seizures is given by

$$E[X] = 0(0.17) + 2(0.21) + 4(0.18) + 6(0.11) + 8(0.16) + 10(0.17)$$

which is equal to (circle one) **4.32** / **4.78** / **5.50** / **5.75**.
(Use your calculator: STAT ENTER; type X, 0, 2, 4, 6 and 8, into L_1 and $p(x)$, 0.17, ... , 0.17, into L_2; then define $L_3 = L_1 \times L_2$; then STAT CALC ENTER 2nd L_3 ENTER; then read $\sum x = 4.78$.)

(c) *General Formula For The Expected Value.* **True** / **False** The general formula for the expected value (mean) is given by

$$E[X] = \sum_{i=1}^{n} x_i p(x_i)$$

2. *Smokers.* The number of smokers, X, in any group of three people is given by the following probability distribution.

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The mean (expected) number of smokers is

$$E[X] = \mu_X = \frac{1}{8}\times 0 + \frac{3}{8}\times 1 + \frac{3}{8}\times 2 + \frac{1}{8}\times 3$$

which is equal to (circle one) **0.5** / **1.5** / **2.5** / **3.5**.

3. *Another Distribution In Tabular Form.* If the distribution is

x	0	1	2	3
$p(x)$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

the mean is

$$E[X] = \frac{4}{8}\times 0 + \frac{2}{8}\times 1 + \frac{1}{8}\times 2 + \frac{1}{8}\times 3 =$$

which is equal to (circle one) **1.500** / **0.875** / **1.375** / **0.625**

4. *Rolling a Pair of Dice.* If the dice are fair, the distribution of X (the sum of two rolls of a pair of dice) is then

x	2	3	4	5	6	7
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$

x	8	9	10	11	12
$P(X = x)$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The mean (expected) sum of the roll of a pair of fair dice is then

$$E[X] = \mu_X = \frac{1}{36}\times 2 + \frac{2}{36}\times 3 + \cdots + \frac{2}{36}\times 11 + \frac{1}{36}\times 12$$

which is equal to (circle one) **5** / **6** / **7** / **8**.
(Think about it: this is a symmetric distribution balanced on what number?)

5. *Expectation and Distribution Function.* If the distribution is

$$p(x) = \frac{3-x}{3},\ x = 1, 2,$$

the mean is

$$E[X] = 1 \times \frac{3-1}{3} + 2 \times \frac{3-2}{3}$$

which is equal to (circle one) $\mathbf{\frac{3}{3}}$ / $\mathbf{\frac{4}{3}}$ / $\mathbf{\frac{5}{3}}$ / $\mathbf{\frac{6}{3}}$

6. *Roulette.* The roulette table has 38 numbers: the numbers are 1 to 36, 0 and 00. A ball is spun on a corresponding roulette wheel which, after a time, settles down and the ball drops into one of 38 slots which correspond to the 38 numbers on the roulette table.

   (a) Let random variable X be the winnings from a \$1 bet placed on an even coming up, where this bet pays 1 to 1. Recall,

x	-\$1	\$1
$p(x)$	$\frac{20}{38}$	$\frac{18}{38}$

   and so the mean is

   $$E[X] = -1 \times \frac{20}{38} + 1 \times \frac{18}{38}$$

   which is equal to (circle one) $-\mathbf{\frac{20}{38}}$ / $-\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{20}{38}}$

   (b) Let random variable X be the winnings from a \$1 bet placed on a section (with 12 numbers) coming up, where this bet pays 2 to 1. Recall,

x	-\$1	\$2
$p(x)$	$\frac{26}{38}$	$\frac{12}{38}$

   and so the mean is

   $$E[X] = -1 \times \frac{26}{38} + 2 \times \frac{12}{38}$$

   which is equal to (circle one) $-\mathbf{\frac{20}{36}}$ / $-\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{20}{38}}$

7. *Binomial Distribution.* The distribution of the binomial random variable is given by

$$p(i) = P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i},\ i = 0, 1, \ldots, n$$

Consider the case when $n = 4$ and $p = 0.25$, where

X	0	1	2	3	4
$p(x)$	0.32	0.42	0.21	0.05	0.004

(a) The mean (expected value) is then

$$E[X] = \mu_X = 0.32\times 0 + 0.42\times 1 + 0.21\times 2 + 0.05\times 3 + 0.004\times 4$$

which is equal to (circle closest one) **1** / **2** / **3** / **4**.

(b) $np =$ (circle closest one) **1** / **2** / **3** / **4**.
(which, notice, is the same answer as above!)

(c) **True** / **False** If X is a binomial random variable, then $E[X] = np$.

8. *Expectation and the Indicator Function.* The random variable I is an indicator function of an event A if

$$I = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{if } A^c \text{ occurs} \end{cases}$$

and so the mean is

$$E[X] = 1 \times P(A) + 0 \times (1 - P(A))$$

which is equal to (circle one) **0** / $\boldsymbol{1 - P(A)}$ / $\boldsymbol{P(A)}$

4.5 Expectation of a Function of a Random Variable

The expected value of a function g of the random variable X, $E[g(X)]$ is given by

$$E[g(X)] = \sum_i g(x_i)p(_i)$$

Exercise 4.5 (Expected Value of a Function of a Discrete Random Variable)

1. *Seizures.* The probability mass function for the number of seizures, X, of a typical epileptic person in any given year is given in the following table.

X	0	2	4	6	8	10
$p(x)$	0.17	0.21	0.18	0.11	0.16	0.17

(a) If the medical costs for each seizure, X, is \$200, $g(x) = 200x$, the new distribution for $g(x)$ becomes,

X	0	2	4	6	8	10
$g(X) = 200x$	$200(0) = 0$	$200(2) = 400$	800	1200	1600	2000
$p(g(x))$	0.17	0.21	0.18	0.11	0.16	0.17

The expected value (mean) cost of seizures is then given by

$$E[g(X)] = E[200X] = [0](0.17) + [400](0.21) + \cdots + [2000](0.17)$$

which is equal to (circle one) **432** / **578** / **750** / **956**.
(Use your calculator: STAT ENTER; type X, 0, 2, 4, 6 and 8, into L_1 and define $g(X)$ in $L_2 = 200 \times L_1$, and type $p(x)$, 0.17, ... , 0.17, into L_3; then define $L_4 = L_2 \times L_3$; then STAT CALC ENTER 2nd L_4 ENTER; then read $\sum x = 956$.)

(b) If the medical costs for each seizure is $g(x) = 200x + 1500$, the new distribution for $g(x)$ becomes,

X	0	2	4	6	8	10
$g(X) = 200x + 1500$	$200(0) + 1500 = 1500$	1900	2300	2700	3100	3500
$p(g(x))$	0.17	0.21	0.18	0.11	0.16	0.17

The expected value (mean) cost of seizures is then given by

$$E[g(X)] = E[200X + 1500] = (1500)(0.17) + (1900)(0.21) + \cdots + (3500)(0.17)$$

which is equal to (circle one) **432** / **578** / **750** / **2456**.

(c) If the medical costs for each seizure is $g(x) = x^2$, the new distribution for $g(x)$ becomes,

X	0	2	4	6	8	10
$g(X) = x^2$	$0^2 = 0$	4	16	36	64	100
$p(g(x))$	0.17	0.21	0.18	0.11	0.16	0.17

The expected value (mean) cost of seizures is then given by

$$E[g(X)] = E[X^2] = (0)(0.17) + (4)(0.21) + \cdots + (100)(0.17)$$

which is equal to (circle one) **34.92** / **57.83** / **75.01** / **94.56**.
($E[X^2]$ is called the *second moment* (about the origin); $E[X^3]$ is called the *third moment*; $E[X^n]$ is called the *nth moment*.))

(d) If the medical costs for each seizure is $g(x) = 200x^2 + x - 5$,
$E[g(X)] = E[200X^2 + X - 5] =$ (circle closest one)
4320 / **5780** / **6983** / **8480**.

2. *Flipping Until a Head Comes Up.* A (weighted) coin has a probability of $p = 0.7$ of coming up heads (and so a probability of $1 - p = 0.3$ of coming up tails). This coin is flipped until a head comes up or until a total of 4 flips are made. Let X be the number of flips. Then, recall,

X	1	2	3	4
$p(x)$	0.7	$0.3(0.7) = 0.21$	$0.3^2(0.7) = 0.063$	$0.3^3 = 0.027$

   (a) $E[X] =$ (circle one) **1.417** / **2.233** / **2.539** / **4.567**
   (b) If $g(x) = 3x + 5$, $E[g(X)] =$ (circle one) **7.417** / **8.233** / **9.251** / **10.567**
   (c) $3E[X] + 5 = 3(1.417) + 5 =$ (circle one) **7.417** / **8.233** / **9.251** / **10.567**
   (d) **True** / **False** $aE[X] + b = E[aX + b]$
   (e) If $g(x) = x^2$ the *second moment* is
   $E[g(X)] = E[X^2] =$ (circle one) **1.539** / **2.233** / **2.539** / **4.567**
   (f) If $g(x) = x^3$ the *third moment* is
   $E[g(X)] = E[X^3] =$ (circle one) **1.539** / **2.233** / **2.539** / **5.809**
   (g) If $g(x) = x^4$ the *fourth moment* is
   $E[g(X)] = E[X^4] =$ (circle one) **11.539** / **12.233** / **12.539** / **16.075**
   In general, $E[X^n]$, is the nth moment.

3. Consider the distribution

$$p(x) = \frac{3-x}{3}, x = 1, 2$$

   (a) The mean is

$$E[X] = [1] \times \frac{3-1}{3} + [2] \times \frac{3-2}{3}$$

   which is equal to (circle one) $\mathbf{\frac{2}{3}}$ / $\mathbf{\frac{3}{3}}$ / $\mathbf{\frac{4}{3}}$ / $\mathbf{\frac{5}{3}}$
   (b) If $g(x) = 3x + 5$,

$$E[g(X)] = E[3X + 5] = 3E[X] + 5 = 3 \times \frac{4}{3} + 5 =$$

   which is equal to (circle one) $\mathbf{\frac{21}{3}}$ / $\mathbf{\frac{22}{3}}$ / $\mathbf{\frac{23}{3}}$ / $\mathbf{\frac{27}{3}}$
   (c) If $g(x) = 6x$,

$$E[g(X)] = E[6X] = 6E[X] = 6 \times \frac{4}{3} =$$

   which is equal to (circle one) $\mathbf{\frac{21}{3}}$ / $\mathbf{\frac{22}{3}}$ / $\mathbf{\frac{23}{3}}$ / $\mathbf{\frac{24}{3}}$

4.6 Variance

We will now look at the *variance*, $V(X)$,

$$\text{Var}(X) = E[(X - \mu)^2]$$

and *standard deviation*, $\text{SD}(X)$, of a random variable, X.

Exercise 4.6 (Standard Deviation of a Discrete Random Variable)

1. *Seizures.* Since the number of seizures, X, of a typical epileptic person in any given year is given by the following probability distribution,

X	0	2	4	6	8	10
P(X = x)	0.17	0.21	0.18	0.11	0.16	0.17

and the expected value (mean) number of seizures is given by $\mu = E(X) = 4.78$,

(a) *A First Look: Variance Measures How "Dispersed" The Distribution Is.*

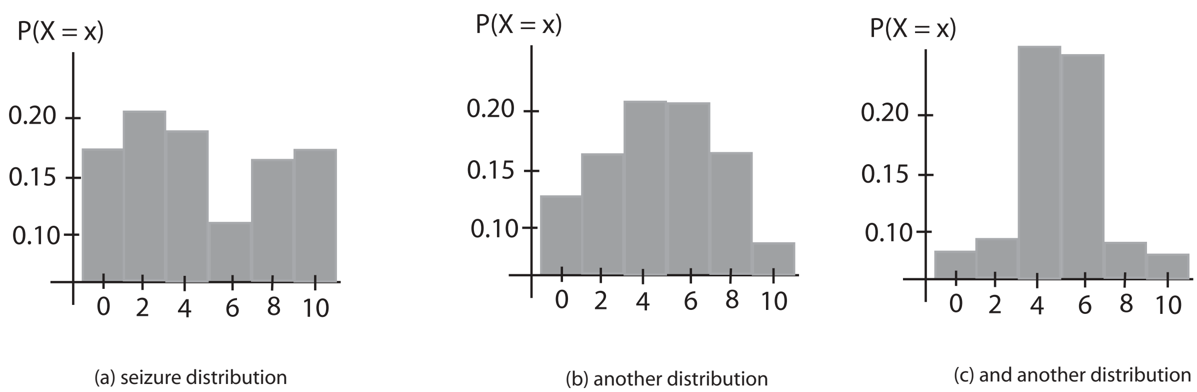


Figure 4.6 (Variance Measures How "Dispersed" The Distribution Is)

If the variance (standard deviation) measures how "spread out" (or "dispersed") the distribution is, then distribution for the number of seizures distribution (a) above, is (circle one) **more** / **as equally** / **less** dispersed than the other two distributions (b) and (c) above.
In other words, if ten (10) is "very" dispersed and zero (0) is not dispersed (concentrated at one point), then the variance for the seizure distribution seems close to (circle one) **0** / **7** / **10**

(b) *Calculating The Variance.* The *variance* is given by,

$$\text{Var}(X) = (0 - 4.78)^2(0.17) + (2 - 4.78)^2(0.21) + \cdots + (10 - 4.78)^2(0.17)$$

which is equal to (circle one) **10.02** / **11.11** / **12.07** / **13.25**.
The *standard deviation* is given by

$$\text{SD}(X) = \sqrt{12.07}$$

which is equal to (circle one) **3.47** / **4.11** / **5.07** / **6.25**.
(Use your calculator: as above, STAT ENTER; type X, 0, 2, 4, 6 and 8, into L_1 and $P(X = x)$, 0.17, ... , 0.17, into L_2; then define $L_3 = (L_1 - 4.78)^2 \times L_2$; then STAT CALC ENTER 2nd L_3 ENTER; then read $\sum x = 12.07$ for the variance; $\sqrt{12.07} = 3.47$ gives the standard deviation.)

(c) If the medical costs for each seizure, X, is \$200, $g(x) = 200x$, the new distribution for $g(x)$ becomes,

X	0	2	4	6	8	10
$g(X) = 200x$	$200(0) = 0$	$200(2) = 400$	800	1200	1600	2000
$p(g(x))$	0.17	0.21	0.18	0.11	0.16	0.17

Since the expected value (mean) cost of seizures is $E[g(X)] = 200E[X] = 200(4.78) = 956$, then the variance for $g(X)$ is given by,

$$\text{Var}(X) = (0 - 0)^2(0.17) + (400 - 956)^2(0.21) + \cdots + (2000 - 956)^2(0.17)$$

which is equal to (circle one) **100200** / **311100** / **4120700** / **482800**.
The *standard deviation* is given by

$$\text{SD}(X) = \sqrt{482800}$$

which is equal to (circle closest one) **347** / **411** / **507** / **695**.

(d) *Using the Formula Var*$(aX + b) = a^2$ *Var*(X). If the medical costs for each seizure, X, is \$200, $g(x) = 200x$, the new distribution for $g(x)$ becomes,

X	0	2	4	6	8	10
$g(X) = 200x$	$200(0) = 0$	$200(2) = 400$	800	1200	1600	2000
$p(g(x))$	0.17	0.21	0.18	0.11	0.16	0.17

Since the variance of X is given by $\text{Var}(X) = 12.07$, the variance of $g(x) = 200x$ is given by

$$\text{Var}[g(X)] = \text{Var}[200X] = 200^2\text{Var}(X) = 200^2(12.07) =$$

100200 / **311100** / **4120700** / **482800**.

2. *Smokers.* Since the number of smokers, X, in any group of three people is given by the following probability distribution.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(a) *One Way To Calculate Variance.* Since the mean (expected) number of smokers is $\mu = 1.5$, then the variance is given by,

$$\text{Var}(X) = (0-1.5)^2\frac{1}{8} + (1-1.5)^2\frac{3}{8} + (2-1.5)^2\frac{3}{8} + (3-1.5)^2\frac{1}{8}$$

which is equal to (circle one) **0.02** / **0.41** / **0.59** / **0.75**.
The standard deviation is given by
$\text{SD}(X) = \sqrt{0.75}$ (circle one) **0.47** / **0.86** / **1.07** / **2.25**.

(b) *Another Way To Calculate Variance.* Since the mean (expected) number of smokers is $E[X] = \mu = 1.5$ and the second moment is given by

$$E[^2] = (0)^2\frac{1}{8} + (1)^2\frac{3}{8} + (2)^2\frac{3}{8} + (3)^2\frac{1}{8} = 3$$

then the variance is given by,

$$\text{Var}(X) = E[X^2] - [E(X)]^2 = 3 - (1.5)^2$$

which is equal to (circle one) **0.02** / **0.41** / **0.59** / **0.75**.
The standard deviation is given by
$\text{SD}(X) = \sqrt{0.75}$ (circle one) **0.47** / **0.86** / **1.07** / **2.25**.

3. *Rolling a Pair of Dice.* If the dice are fair, the distribution of X (the sum of two rolls of a pair of dice) is

x	2	3	4	5	6	7
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$

x	8	9	10	11	12
$P(X = x)$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

where, remember, the expected value is 7 and so

$$\text{Var}(X) = (2-7)^2\frac{1}{36} + (3-7)^2\frac{2}{36} + \cdots + (11-7)^2\frac{1}{36} + (12-7)^2\frac{1}{36}$$

which is equal to (circle one) $\mathbf{\frac{34}{6}}$ / $\mathbf{\frac{35}{6}}$ / $\mathbf{\frac{36}{6}}$.
The standard deviation is given by
$\text{SD}(X) = \sqrt{\frac{35}{6}}$ (circle one) **0.47** / **0.86** / **1.07** / **2.42**.

4. *And Yet Another Distribution.* Since the distribution is

$$P(X = x) = \frac{3-x}{3}, x = 1, 2,$$

and $\mu = \frac{4}{3}$, then

$$\text{Var}(X) = \left(1 - \frac{4}{3}\right)^2 \frac{3-1}{3} + \left(2 - \frac{4}{3}\right)^2 \frac{3-2}{3},$$

which is equal to (circle one) $\mathbf{\frac{2}{9}}$ / $\mathbf{\frac{3}{9}}$ / $\mathbf{\frac{4}{9}}$ / $\mathbf{\frac{5}{9}}$.
Also, $\text{SD}(X) = \sqrt{\frac{2}{9}}$ = (circle one) **0.05** / **0.47** / **1.07** / **2.25**.

5. *Roulette.*

   (a) Let random variable X be the winnings from a \$1 bet placed on an even coming up, where this bet pays 1 to 1. Recall,

x	-\$1	\$1
$p(x)$	$\frac{20}{38}$	$\frac{18}{38}$

   where the mean is $\mu = -\frac{2}{38}$ and so

   $$\text{Var}(X) = \left(-1 - \left(-\frac{2}{38}\right)\right)^2 \frac{20}{38} + \left(1 - \left(-\frac{2}{38}\right)\right)^2 \frac{18}{38},$$

   which is equal to (circle one) $\mathbf{\frac{360}{361}}$ / $\mathbf{\frac{860}{361}}$ / $\mathbf{\frac{891}{361}}$ / $\mathbf{\frac{932}{361}}$.
   Also, $\text{SD}(X) = \sqrt{\frac{2}{9}}$ = (circle one) **0.051** / **0.999** / **1.573** / **2.251**.

   (b) Let random variable X be the winnings from a \$1 bet placed on a section (with 12 numbers) coming up, where this bet pays 2 to 1. Recall,

x	-\$1	\$2
$p(x)$	$\frac{26}{38}$	$\frac{12}{38}$

   where the mean is $\mu = -\frac{2}{38}$ and so

   $$\text{Var}(X) = \left(-1 - \left(-\frac{2}{38}\right)\right)^2 \frac{26}{38} + \left(2 - \left(-\frac{2}{38}\right)\right)^2 \frac{12}{38},$$

   which is equal to (circle one) $\mathbf{\frac{702}{361}}$ / $\mathbf{\frac{860}{361}}$ / $\mathbf{\frac{891}{361}}$ / $\mathbf{\frac{932}{361}}$.
   Also, $\text{SD}(X) = \sqrt{\frac{2}{9}}$ = (circle one) **0.05** / **0.47** / **1.39** / **2.25**.

6. *Mathematical Manipulations Of Variance and Expectation.*

   (a) If $E[X] = 4$ and $\text{Var}(X) = 3$, then
   $E[5X - 2] = 5E[X] - 2 = 5(4) - 2 =$ (circle one) **16** / **18** / **20**.
   $\text{Var}[5X - 2] = 5^2\text{Var}(X) = 25(3) =$ (circle one) **15** / **18** / **75**.
   $E[7X + 5] = 7E[X] + 5 = 7(4) + 5 =$ (circle one) **16** / **28** / **33**.
   $\text{Var}[7X + 5] = 7^2\text{Var}(X) = 49(3) =$ (circle one) **134** / **118** / **147**.

(b) If $E[X] = -2$ and $\text{Var}(X) = 6$, then
$E[5X - 2] = 5E[X] - 2 = 5(-2) - 2 =$ (circle one) $\mathbf{-12}$ / $\mathbf{-18}$ / $\mathbf{-20}$.
$\text{Var}[5X - 2] = 5^2\text{Var}(X) = 25(6) =$ (circle one) **150** / **180** / **750**.
$E[7X + 5] = 7E[X] + 5 = 7(-2) + 5 =$ (circle one) $\mathbf{-7}$ / $\mathbf{-9}$ / $\mathbf{-11}$.
$\text{Var}[-7X + 5] = (-7)^2\text{Var}(X) = 49(6) =$ (circle one) **234** / **268** / **294**.

(c) If $E[X] = -2$ and $\text{Var}(X) = 6$, then
$E[X^2] = E[X] -$ $\text{Var}(X) = -2 - (6) =$ (circle one) $\mathbf{-6}$ / $\mathbf{-8}$ / $\mathbf{-20}$.

Review Chapter 5

Continuous Random Variables

5.1 Introduction

For all real $x \in (-\infty, \infty)$,

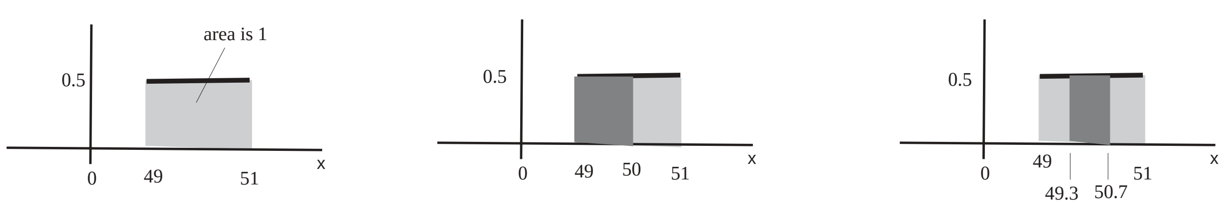
$$P\{X \in B\} = \int_B f(x)\,dx$$

where $f(x)$ is called the *probability density function* and so

$$\begin{aligned} P\{x \leq X \leq B\} &= \int_a^b f(x)\,dx \\ P\{X = a\} &= \int_a^a f(x)\,dx \; = 0 \\ P\{X < a\}\; P\{X \leq a\} &= \int_{-\infty}^a f(x)\,dx \end{aligned}$$

Exercise 5.1 (Introduction to Continuous Random Variables)

1. *A First Look: Uniform Probability Distribution and Potatoes* An automated process fills one bag after another with Idaho potatoes. Although each filled bag should weigh 50 pounds, in fact, because of the differing shapes and weights of each potato, each bag weighs anywhere from 49 pounds to 51 pounds, as indicated in the three graphs below.



(a) f(x) = 0.5 on [49, 51]

(b) f(x) = 0.5 on [49, 51]

(c) f(x) = 0.5 on [49, 51]

Figure 5.1 (Uniform Distributions and Potatoes)

(a) If all of the filled bags *must* fall between 49 and 51 pounds, then there is (circle one) **a little** / **no** chance that one filled bag, chosen at random from all filled bags, will weigh 48.5 pounds.

(b) There is (circle one) **a little** / **no** chance that one filled bag, chosen at random, will weigh 51.5 pounds.

(c) There is a (circle one) **100%** / **50%** / **0%** chance that one randomly chosen filled bag chosen will weigh 53.5 pounds.

(d) One randomly chosen filled bag will weigh 36 pounds with probability (circle one) **1** / **0.5** / **0**.

(e) One randomly chosen filled bag will weigh *(strictly) less than 49 pounds* with probability $P(x < 49) =$ (circle one) **1** / **0.5** / **0**.

(f) One randomly chosen filled bag will weigh *(strictly) more than 51 pounds* with probability $P(x > 51) =$ (circle one) **1** / **0.5** / **0**.

(g) *Figure (a).* One randomly chosen filled bag will weigh *between 49 and 51 pounds (inclusive)* with probability $P(49 \leq x \leq 51) =$
(circle one) **1** / **0.5** / **0**.

(h) *More Figure (a).* The probability $P(49 \leq x \leq 51)$ is represented by or equal to the (circle none, one or more)

   i. rectangular area equal to 1.
   ii. rectangular area equal to the width $(51 - 49 = 2)$ times the height (0.5).
   iii. definite integral of $f(x) = 0.5$ over the interval $[49, 51]$.
   iv. $\int_{49}^{51} 0.5\, dx = [0.5x]_{49}^{51} = 0.5(51) - 0.5(49) = 1$.

(i) **True** / **False** The *probability density function* is given by the piecewise function,

$$f(x) = \begin{cases} 0 & \text{if } x < 49 \\ 0.5 & \text{if } 49 \leq x \leq 51 \\ 0 & \text{if } x > 51 \end{cases}$$

This is an example of a *uniform* probability density function (pdf).

(j) *Figure (b).* One randomly chosen filled bag will weigh *between 49 and 50 (not 51!) pounds (inclusive)* with probability
$P(49 \leq x \leq 50) = (50 - 49)(0.5) =$ (circle one) **0** / **0.5** / **1**.

(k) *More Figure (b).* One randomly chosen filled bag will weigh between 49 and 50 pounds (inclusive) with probability
$P(49 \leq x \leq 50) = \int_{49}^{50} 0.5\, dx = [0.5x]_{49}^{50} = 0.5(50) - 0.5(49) =$
(circle one) **0** / **0.5** / **1**.

(l) *Figure (c).* One randomly chosen filled bag will weigh between 49.3 and 50.7 pounds (inclusive) with probability
$P(49.3 \leq x \leq 50.7) = (50.7 - 49.3)(0.5) =$ (circle one) **0** / **0.5** / **0.7**.

(m) *More Figure (c).* One randomly chosen filled bag will weigh between 49.3 and 50.7 pounds (inclusive) with probability
$P(49.3 \leq x \leq 50.7) = \int_{49.3}^{50.7} 0.5\,dx = [0.5x]_{49.3}^{50.7} = 0.5(50.7) - 0.5(49.3) =$
(circle one) **0** / **0.5** / **0.7**.

(n) $P(49.1 \leq x \leq 50.9) = \int_{49.1}^{50.9} 0.5\,dx = [0.5x]_{49.1}^{50.9} = 0.5(50.9) - 0.5(49.1) =$
(circle one) **0** / **0.5** / **0.9**.

(o) Another example.

$$\begin{aligned} P(x \leq 50.9) &= \int_{-\infty}^{50.9} f(x)\,dx \\ &= \int_{-\infty}^{49} f(x)\,dx + \int_{49}^{50.9} f(x)\,dx \\ &= \int_{-\infty}^{49} 0\,dx + \int_{49}^{50.9} 0.5\,dx \\ &= 0 + [0.5x]_{49}^{50.9} \\ &= 0.5(50.9) - 0.5(49) \ = \end{aligned}$$

(circle one) **0** / **0.5** / **0.95**.

(p) Another example.

$$\begin{aligned} P(x \leq 50.2) &= \int_{-\infty}^{50.2} f(x)\,dx \\ &= \int_{-\infty}^{49} f(x)\,dx + \int_{49}^{50.2} f(x)\,dx \\ &= \int_{-\infty}^{49} 0\,dx + \int_{49}^{50.2} 0.5\,dx \\ &= 0 + [0.5x]_{49}^{50.2} \\ &= 0.5(50.2) - 0.5(49) \ = \end{aligned}$$

(circle one) **0** / **0.6** / **0.95**.

(q) Another example.

$$\begin{aligned} P(x \geq 50.2) &= \int_{50.2}^{\infty} f(x)\,dx \\ &= \int_{50.2}^{51} f(x)\,dx + \int_{51}^{\infty} f(x)\,dx \\ &= \int_{50.2}^{51} 0.5\,dx + \int_{51}^{\infty} 0\,dx \\ &= [0.5x]_{50.2}^{51} + 0 \\ &= 0.5(51) - 0.5(50.2) \;= \end{aligned}$$

(circle one) **0.4** / **0.6** / **0.95**.

2. *More Probability Density Distributions.* In addition to the uniform probability density function, there are other probability density functions, as shown in the three graphs below.

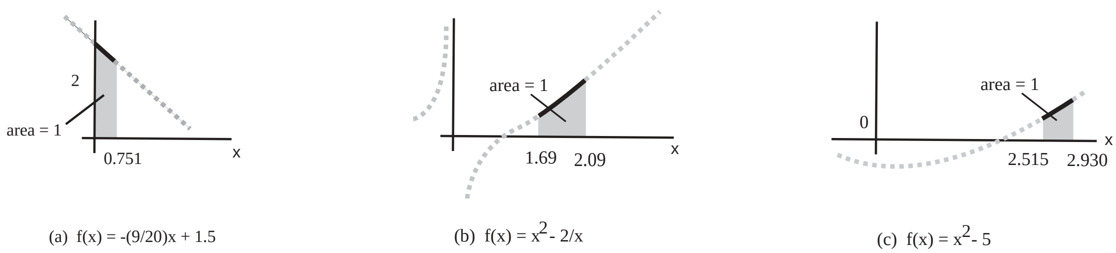


Figure 5.2 (Different Probability Density Functions)

(a) *Figure (a),* $f(x) = -\frac{9}{20}x + 1.5$ *on* $[0, 0.751]$. The probability $P([0, 0.751])$ is represented by or equal to the (circle none, one or more)

i. shaded area equal to 1.

ii. definite integral of $f(x) = -\frac{9}{20}x + 1.5$ defined over the interval $[0, 0.751]$.

iii. $\int_0^{0.751} \left(-\frac{9}{20}x + 1.5\right) dx = \left[-\frac{9}{40}x^2 + 1.5x\right]_0^{0.751} = 1$

($Y_1 = -\frac{9}{20}x + 1.5$, WINDOW 0 4 1 0 2 1, GRAPH, 2nd CALC 7:$\int f(x)\,dx$)

(b) *More Figure (a).* **True** / **False** The probability density function is given by the piecewise function,

$$f(x) = \begin{cases} -\frac{9}{20}x + 1.5 & \text{if } 0 \leq x \leq 0.751 \\ 0 & \text{elsewhere} \end{cases}$$

(c) *More Figure (a)*

$$\begin{aligned} P(0.1 \le x \le 0.5) &= \int_{0.1}^{0.5} \left(-\frac{9}{20}x + 1.5\right) dx \\ &= \left[-\frac{9}{40}x^2 + 1.5x\right]_{0.1}^{0.5} = \end{aligned}$$

(circle one) **0.446** / **0.546** / **0.646**.

(d) *More Figure (a).* $P(x \ge 0.4) =$ (circle one) **0.436** / **0.546** / **0.646**.

(e) *Figure (b),* $f(x) = x^2 - \frac{2}{x}$ *on* $[1.69, 2.09]$. The probability $P([1.69, 2.09])$ is represented by or equal to the (circle none, one or more)

i. shaded area equal to 1.

ii. definite integral of $f(x) = x^2 - \frac{2}{x}$ defined over the interval $[1.69, 2.09]$.

iii. $\int_{1.69}^{2.09} \left(x^2 - \frac{2}{x}\right) dx = \left[\frac{1}{3}x^3 - 2\ln x\right]_{1.69}^{2.09} = 1$

($\mathrm{Y}_1 = x^2 - \frac{2}{x}$,

(f) *More Figure (b).* **True** / **False** The probability density function is given by the piecewise function,

$$f(x) = \begin{cases} x^2 - \frac{2}{x} & \text{if } 1.69 \le x \le 2.09 \\ 0 & \text{elsewhere} \end{cases}$$

(g) *More Figure (b).* The following piecewise function,

$$f(x) = \begin{cases} x^2 - \frac{2}{x} & \text{if } 3 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

(circle one) **is** / **is not** a probability density function because $\int_3^5 \left(x^2 - \frac{2}{x}\right) dx \ne 1$

(h) *More Figure (b).* The following piecewise function,

$$f(x) = \begin{cases} x^2 - \frac{2}{x} & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

is *not* a probability density function because (circle *two*)

i. it is *not* continuous (there are "gaps" in the interval).

ii. negative.

iii. its integral the interval $[0, 1]$ does *not* exactly equal 1.

(i) *More Figure (b).*$P(0.1 \le x \le 0.5) =$ (circle one) **0** / **0.546** / **0.646**.

(j) *More Figure (b).* $P(x \ge 0.4) =$ (circle one) **0.436** / **0.546** / **1**.

(k) *Figure (c),* $f(x) = x^2 - 5$. The function $f(x) = x^2 - 5$ is a probability density function if defined on the interval (circle one) $\mathbf{[0.001, 0.251]}$ / $\mathbf{[2.515, 2.930]}$ / $\mathbf{[1.545, 1.978]}$

(l) *More Figure (c)*

$$\begin{aligned} P(2.6 \le x \le 2.7) &= \int_{2.6}^{2.7} \left(x^2 - 5\right) dx \\ &= \left[\frac{1}{3}x^3 - 5x\right]_{2.6}^{2.7} = \end{aligned}$$

(circle one) **0.202** / **0.546** / **0.646**.

(m) *More Figure (c).* $P(x \ge 0.4) =$ (circle one) **0.436** / **0.546** / **1**.

3. *Normalizing Continuous Functions Into Probability Density Functions*

(a) Find C such that $f(x) = Cx$ is a probability density function over the interval $[2, 4]$. In other words, find C such that

$$\begin{aligned} P(2 \le x \le 4) &= \int_2^4 Cx\,dx \\ &= \left[\frac{C}{2}x^2\right]_2^4 \\ &= \frac{C}{2}(4)^2 - \frac{C}{2}(2)^2 \\ &= \frac{C}{2}\left((4)^2 - (2)^2\right) \\ &= \frac{C}{2}(12) \\ &= 6C \\ &= 1 \end{aligned}$$

and so $C =$ (circle one) $\mathbf{\frac{1}{4}}$ / $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{1}{6}}$.

$\int_2^4 \frac{1}{6}x\,dx = 1.$

(b) Find k such that $f(x) = kx$ is a probability density function over the interval $[1,5]$. In other words, find k such that

$$\begin{aligned} P(1 \leq x \leq 5) &= \int_1^5 kx \, dx \\ &= \left[\frac{k}{2}x^2\right]_1^5 \\ &= \frac{k}{2}(5)^2 - \frac{k}{2}(1)^2 \\ &= \frac{k}{2}\left((5)^2 - (1)^2\right) \\ &= \frac{k}{2}(24) \\ &= 12k \\ &= 1 \end{aligned}$$

and so $k =$ (circle one) $\mathbf{\frac{1}{4}}$ / $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{1}{12}}$.

(c) Find k such that $f(x) = kx^2$ is a probability density function over the interval $[1,5]$. In other words, find k such that

$$\begin{aligned} P(1 \leq x \leq 5) &= \int_1^5 kx^2 \, dx \\ &= \left[\frac{k}{3}x^3\right]_1^5 \\ &= \frac{k}{3}(5)^3 - \frac{k}{3}(1)^3 \\ &= \frac{k}{3}\left((5)^3 - (1)^3\right) \\ &= \frac{k}{3}(124) \\ &= \frac{124}{3}k \\ &= 1 \end{aligned}$$

and so $k =$ (circle one) $\mathbf{\frac{3}{26}}$ / $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{3}{124}}$.

(d) Find k such that $f(x) = k(x-3)$ is a probability density function over the interval $[1, 5]$. In other words, find k such that

$$\begin{aligned} P(1 \leq x \leq 5) &= \int_1^5 k(x+3)\,dx \\ &= \left[\frac{k}{2}x^2 + 3kx\right]_1^5 \\ &= \left(\frac{k}{2}(5)^2 + 3k(5)\right) - \left(\frac{k}{2}(1)^2 + 3k(1)\right) \\ &= \frac{k}{2}(25-1) + k(15-3) \\ &= \frac{k}{2}(24) + 12k \\ &= 24k \\ &= 1 \end{aligned}$$

and so $k =$ (circle one) $\mathbf{\frac{3}{26}}$ / $\mathbf{\frac{1}{24}}$ / $\mathbf{\frac{1}{18}}$.

(e) *Exponential Distribution and Improper Integration.* $\int_0^\infty \lambda e^{-\lambda x}\,dx =$ (circle none, one or more)
$\boldsymbol{\lim_{b\to\infty} \left[\frac{\lambda}{-\lambda} e^{-\lambda x}\right]_0^b}$
$\boldsymbol{\lim_{b\to\infty} \left[-e^{-\lambda b} - \left(-e^{-\lambda(0)}\right)\right]}$
$\boldsymbol{\lim_{b\to\infty} \left[-e^{-\lambda b} + 1\right]}$
and so $\int_0^\infty \lambda e^{-\lambda x}\,dx =$ (circle one) $\mathbf{-1}$ / $\mathbf{0}$ / $\mathbf{1}$
and so $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$ is a probability density function

5.2 Expectation and Variance of Continuous Random Variables

Let $f(x)$ be the probability density function. Then, the expected value, denoted $E[X]$, (or μ) is defined as

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

and the variance, denoted $\text{Var}(X)$, is defined as

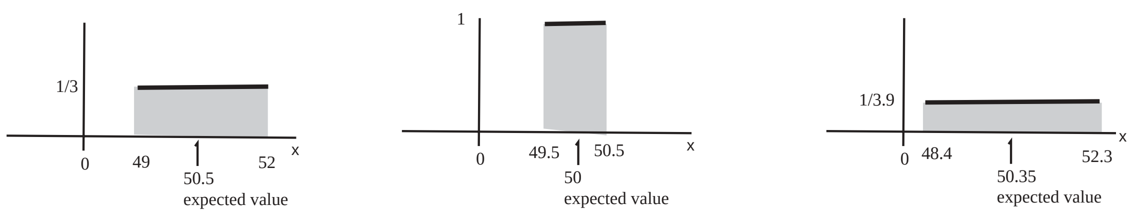
$$\text{Var}(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

and the standard deviation is defined as the square root of the variance. Some properties include,

$$
\begin{aligned}
E[g(X)] &= \int_{-\infty}^{\infty} g(x)f(x)\,dx \\
E[aX+b] &= aE[X]+b \\
\mathrm{Var}(X) &= a^2\mathrm{Var}(X)
\end{aligned}
$$

Exercise 5.2 (Expected Value, Variance and Standard Deviation)

1. *Expected Values For Uniform Probability Density Functions: Potatoes Again* Consider, again, the different automated processes which fill bags of Idaho potatoes which have different uniform probability density functions, as shown in the three graphs below.



(a) f(x) = 1/3 on [49, 52] (b) f(x) = 1 on [49.5, 50.5] (c) f(x) = 1/3.9 on [48.4, 52.3]

Figure 5.3 (Expected Values of Uniform Probability Density Functions)

(a) *Figure (a).* Since the weight of potatoes are *uniformly* spread over the interval $[49, 52]$, we would *expect* the weight of a potato chosen at random from all these potatoes to be
$\frac{49+52}{2}$ = (circle one) **50** / **50.5** / **51**.

(b) *Figure (a) Again.* The *expected* weight of a potato chosen at random can also be calculated in the following way:

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf(x)\,dx \\
&= \int_{-\infty}^{49} xf(x)\,dx + \int_{49}^{52} xf(x)\,dx + \int_{52}^{\infty} xf(x)\,dx \\
&= \int_{-\infty}^{49} x(0)\,dx + \int_{49}^{52} x\frac{1}{3}\,dx + \int_{52}^{\infty} x(0)\,dx \\
&= 0 + \left[\frac{1}{6}x^2\right]_{49}^{52} + 0 \\
&= \frac{1}{6}(52)^2 - \frac{1}{6}(49)^2 \; =
\end{aligned}
$$

50 / **50.5** / **51**.

(c) *Figure (b).* Since the weight of potatoes are *uniformly* spread over the interval $[49.5, 50.5]$, we would *expect* the weight of a potato chosen at random from all these potatoes to be
$\frac{49.5+50.5}{2}$ = (circle one) **50** / **50.5** / **51**.

(d) *Figure (b) Again.* The expected weight of a potato chosen at random can also be calculated in the following way:

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f(x)\,dx \\
&= \int_{-\infty}^{49.5} x f(x)\,dx + \int_{49.5}^{50.5} x f(x)\,dx + \int_{50.5}^{\infty} x f(x)\,dx \\
&= \int_{-\infty}^{49.5} x(0)\,dx + \int_{49.5}^{50.5} x(1)\,dx + \int_{50.5}^{\infty} x(0)\,dx \\
&= 0 + \left[\frac{1}{2}x^2\right]_{49.5}^{50.5} + 0 \\
&= \frac{1}{2}(50.5)^2 - \frac{1}{2}(49.5)^2 \; =
\end{aligned}$$

50 / **50.5** / **51**.

(e) *Figure (c).* The expected weight of a potato chosen at random can also be calculated in the following way:

$$\begin{aligned}
E(x) &= \int_{-\infty}^{\infty} x f(x)\,dx \\
&= \int_{48.4}^{52.3} x \frac{1}{3.9}\,dx \\
&= \left[\frac{1}{7.8}x^2\right]_{48.4}^{52.3} \\
&= \frac{1}{2(7.8)}(52.3)^2 - \frac{1}{2(7.8)}(48.4)^2 \; =
\end{aligned}$$

50.15 / **50.35** / **51.15**.

2. *Expected Values For Other Probability Density Functions* Consider the following probability density functions, as shown in the three graphs below.

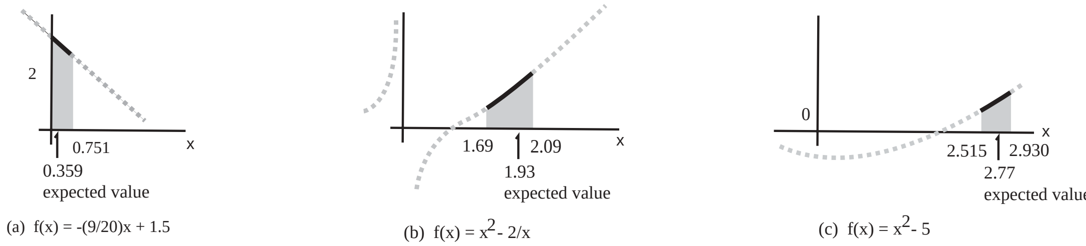


Figure 5.4 (Expected Values of Other Probability Density Functions)

(a) *Figure (a).* Since there is "more" probability on the "left" of the interval $[0, 0.751]$, we would *expect* the *expected* (or mean) weight of value chosen from this distribution to be (circle one) **smaller than / equal to / larger than** the middle value, $\frac{0+0.751}{2} = 0.3755$.

(b) *Figure (a) Again.* The expected value is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{0.751} x f(x)\,dx + \int_{0.751}^{\infty} x f(x)\,dx \\ &= \int_{-\infty}^{0} x(0)\,dx + \int_{0}^{0.751} x\left(-\frac{9}{20}x + 1.5\right)\,dx + \int_{0.751}^{\infty} x(0)\,dx \\ &= \int_{0}^{0.751} \left(-\frac{9}{20}x^2 + 1.5x\right)\,dx \\ &= \left[\frac{-9}{60}x^3 + \frac{1.5}{2}x^2\right]_0^{0.751} = \end{aligned}$$

0.359 / 0.376 / 0.410.
(Use MATH 9:fnInt for $\int_0^{0.751}\left(-\frac{9}{20}x^2 + 1.5x\right)\,dx$.)

(c) *Figure (b).* Since there is "more" probability on the "right" of the interval $[1.69, 2.09]$, we would expect the expected (or mean) weight of value chosen from this distribution to be (circle one) **smaller than / equal to / larger than** the middle value, $\frac{1.69+2.09}{2} = 1.89$.

(d) *Figure (b) Again.* The expected value is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_{1.69}^{2.09} x\left(x^2 - \frac{2}{x}\right)\,dx \\ &= \int_{1.69}^{2.09} \left(x^3 - 2\right)\,dx = \end{aligned}$$

1.89 / 1.93 / 2.04.

(e) *Figure (c).* The expected value is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_{2.515}^{2.930} x\left(x^2-5\right)\, dx \\ &= \int_{2.515}^{2.930} \left(x^3-5x\right)\, dx \;= \end{aligned}$$

(circle one) **2.77** / **2.93** / **3.04**.

3. *Expected Values For Different Functions, $E[g(x)]$.*

(a) The expected value of $g(X) = X + 2$ where the probability density is $f(x) = x^2 - 5$ on the interval $[2.515, 2.930]$, is

$$\begin{aligned} E(X+2) &= \int_{-\infty}^{\infty} (x+2) f(x)\, dx \\ &= \int_{2.515}^{2.930} (x+2)\left(x^2-5\right)\, dx \\ &= \int_{2.515}^{2.930} \left(x^3+2x^2-5x-10\right)\, dx \;= \end{aligned}$$

(circle one) **2.77** / **2.93** / **4.79**.

(b) The expected value of $g(X) = X^2$ where the probability density is $f(x) = x^2 - 5$ on the interval $[2.515, 2.930]$, is

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx \\ &= \int_{2.515}^{2.930} x^2\left(x^2-5\right)\, dx \\ &= \int_{2.515}^{2.930} \left(x^4-5x^2\right)\, dx \;= \end{aligned}$$

(circle one) **6.77** / **7.65** / **8.79**.

(c) The expected value of $g(X) = 2X^3$ where the probability density is $f(x) = x^2 - 5$ on the interval $[2.515, 2.930]$, is

$$\begin{aligned} E[2X^3] &= \int_{-\infty}^{\infty} 2x^3 f(x)\, dx \\ &= \int_{2.515}^{2.930} 2x^3\left(x^2-5\right)\, dx \\ &= \int_{2.515}^{2.930} \left(2x^5-10x^3\right)\, dx \;= \end{aligned}$$

(circle one) **36.77** / **37.65** / **42.32**.

(d) Consider the probability density $f(x) = x^2 - 5$ on the interval $[2.515, 2.930]$. Then,

$$
\begin{aligned}
E[2X^3] - 2 &= \int_{-\infty}^{\infty} 2x^3 f(x)\,dx - 2 \\
&= \int_{2.515}^{2.930} 2x^3\left(x^2 - 5\right)\,dx - 2 \\
&= \int_{2.515}^{2.930} \left(2x^5 - 10x^3\right)\,dx - 2 \;=
\end{aligned}
$$

(circle one) **36.77** / **40.32** / **42.32**.

(e) Consider the probability density $f(x) = x^2 - 5$ on the interval $[2.515, 2.930]$. Then,

$$
\begin{aligned}
E[X^2] - 3E[X] &= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 3\int_{-\infty}^{\infty} x f(x)\,dx \\
&= \int_{2.515}^{2.930} x^2\left(x^2 - 5\right)\,dx - 3\int_{2.515}^{2.930} x\left(x^2 - 5\right)\,dx \\
&= \int_{2.515}^{2.930} \left(x^4 - 5x^2\right)\,dx - 3\int_{2.515}^{2.930} \left(x^3 - 5x\right)\,dx \;=
\end{aligned}
$$

(circle one) **−0.667** / **0.667** / **2.322**.

4. *Variance (and Standard Deviation) For Different Probability Density Functions.*

(a) The variance of the probability density $f(x) = -\frac{9}{20}x + 1.5$ on the interval $[0, 0.751]$, is

$$
\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - [E(X)]^2 \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2 \\
&= \int_{0}^{0.751} x^2\left(-\frac{9}{20}x + 1.5\right)\,dx - \left[\int_{0}^{0.751} x\left(-\frac{9}{20}x + 1.5\right)\,dx\right]^2 \\
&= \int_{0}^{0.751} \left(-\frac{9}{20}x^3 + 1.5x^2\right)\,dx - \left[\int_{0}^{0.751} \left(-\frac{9}{20}x^2 + 1.5x\right)\,dx\right]^2
\end{aligned}
$$

(circle one) **0.173** / **0.047** / **0.123**.
The *standard deviation*, then is $\sigma = \sqrt{0.047} =$
(circle one) **0.173** / **0.047** / **0.216**.

(b) The variance of the probability density $f(x) = x^2 - \frac{2}{x}$ on the interval $[1.69, 2.09]$, is

$$\begin{aligned}
\mathrm{Var}(X) &= E(x^2) - [E(x)]^2 \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2 \\
&= \int_{1.69}^{2.09} x^2\left(x^2 - \frac{2}{x}\right)\,dx - \left[\int_{1.69}^{2.09} x\left(x^2 - \frac{2}{x}\right)\,dx\right]^2 \\
&= \int_{1.69}^{2.09} \left(x^4 - 2x\right)\,dx - \left[\int_{1.69}^{2.09} \left(x^3 - 2\right)\,dx\right]^2 =
\end{aligned}$$

(circle one) **−0.02** / **0.02** / **0.04**
(which is incorrect, due to round off error in the calculator)
The standard deviation (assuming a variance of 0.02), is $\sigma = \sqrt{0.02} =$ (circle one) **0.141** / **0.047** / **0.216**.

(c) The variance of the probability density $f(x) = x^2 - 5$ on the interval $[2.515, 2.930]$, is

$$\begin{aligned}
\mathrm{Var}(X) &= E(x^2) - [E(x)]^2 \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2 \\
&= \int_{2.515}^{2.930} x^2\left(x^2 - 5\right)\,dx - \left[\int_{2.515}^{2.930} x\left(x^2 - 5\right)\,dx\right]^2 \\
&= \int_{2.515}^{2.930} \left(x^4 - 5x^2\right)\,dx - \left[\int_{2.515}^{2.930} \left(x^3 - 5x\right)\,dx\right]^2 =
\end{aligned}$$

(circle one) **−0.04** / **0.04** / **0.04**
(which is also incorrect, due to round off error in the calculator)
The standard deviation (assuming a variance of 0.04), is $\sigma = \sqrt{0.04} =$ (circle one) **0.146** / **0.199** / **0.216**.

(d) **True** / **False**. The variance (and standard deviation) provide a measure of how "spread out" of "dispersed" the probability density function is from the expected value.

5.3 The Uniform Random Variable

Uniform random variable X has a probability density function on the interval (α, β) where

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{elsewhere} \end{cases}$$

and a distribution function,

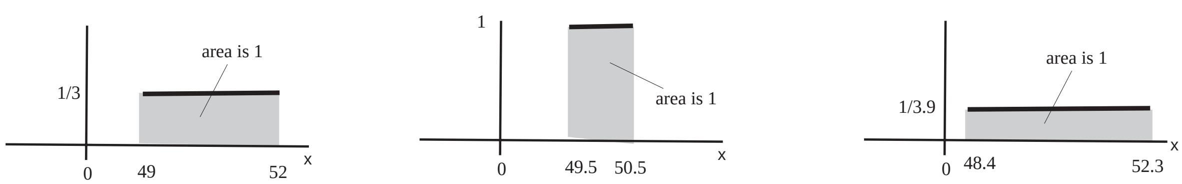
$$F(a) = \begin{cases} 0 & \text{if } a \leq \alpha \\ \frac{a-\alpha}{\beta-\alpha} & \text{if } \alpha < a < \beta \\ 1 & \text{if } a \geq \beta \end{cases}$$

where the expected value and variance are

$$\begin{aligned} E[X] &= \frac{\beta+\alpha}{2} \\ \text{Var}(X) &= \frac{(\beta-\alpha)^2}{12} \end{aligned}$$

Exercise 5.3 (Uniform Random Variable)

1. *Uniform Probability Density Functions: Potatoes Again* Different automated processes which fill bags of Idaho potatoes have different uniform probability density functions, as shown in the three graphs below.



(a) f(x) = 1/3 on [49, 52] (b) f(x) = 1 on [49.5, 50.5] (c) f(x) = 1/3.9 on [48.4, 52.3]

Figure 5.5 (Uniform Probability Density Functions and Potatoes)

   (a) *Figure (a).*

      i. The probability $P(49 \leq x \leq 52)$ is represented by or equal to the (circle none, one or more)

         A. rectangular area equal to 1.

         B. rectangular area equal to the width $(52-49 = 1)$ times the height $(\frac{1}{3})$.

         C. definite integral of $f(x) = \frac{1}{3}$ over the interval $[49, 52]$.

         D. $\int_{49}^{52} \frac{1}{3}\, dx = \left[\frac{1}{3}x\right]_{49}^{52} = \frac{1}{3}(52) - \frac{1}{3}(49) = 1$.

      ii. **True** / **False** The *probability density function* is given by the piecewise function,

$$f(x) = \begin{cases} 0 & \text{if } x < 49 \\ \frac{1}{3} & \text{if } 49 \leq x \leq 52 \\ 0 & \text{if } x > 52 \end{cases}$$

iii. Probability By Integrating.

$$\begin{aligned}
P(x \leq 50.2) &= \int_{-\infty}^{50.2} f(x)\,dx \\
&= \int_{-\infty}^{49} f(x)\,dx + \int_{49}^{50.2} f(x)\,dx \\
&= \int_{-\infty}^{49} 0\,dx + \int_{49}^{50.2} \frac{1}{3}\,dx \\
&= 0 + \left[\frac{1}{3}x\right]_{49}^{50.2} \\
&= \frac{1}{3}(50.2) - \frac{1}{3}(49) =
\end{aligned}$$

(circle one) $\mathbf{\frac{1}{3}}$ / $\mathbf{\frac{1.2}{3}}$ / $\mathbf{\frac{1.4}{3}}$.

iv. Probability By Distribution Function.

$$\begin{aligned}
P(x \leq 50.2) &= F(50.2) \\
&= \frac{50.2 - \alpha}{\beta - \alpha} \\
&= \frac{50.2 - 49}{52 - 49} =
\end{aligned}$$

(circle one) $\mathbf{\frac{1}{3}}$ / $\mathbf{\frac{1.2}{3}}$ / $\mathbf{\frac{1.4}{3}}$.

v. $P(x \geq 50.2) =$ (circle one) $\mathbf{\frac{0.8}{3}}$ / $\mathbf{\frac{1.2}{3}}$ / $\mathbf{\frac{1.8}{3}}$.

vi. *Expectation and Variance.*
$E[X] = \frac{\beta+\alpha}{2} = \frac{52+49}{2} =$ (circle one) **50** / **50.5** / **51**.
$\text{Var}(X) = \frac{(\beta-\alpha)^2}{12} = \frac{(52-49)^2}{12}$ (circle one) **0.75** / **1** / **1.25**.

(b) *Figure (b).*

i. The probability $P(49.5 \leq x \leq 50.5)$ is represented by or equal to the (circle none, one or more)

A. rectangular area equal to 1.

B. rectangular area equal to the width $(50.5 - 49.5 = 3)$ times the height (1).

C. definite integral of $f(x) = 1$ over the interval $[49.5, 50.5]$.

D. $\int_{49.5}^{50.5} 1\,dx = [x]_{49.5}^{50.5} = 51.5 - 49.5 = 1$.

ii. **True** / **False** The probability density function is given by the piecewise function,

$$f(x) = \begin{cases} 1 & \text{if } 49.5 \leq x \leq 50.5 \\ 0 & \text{elsewhere} \end{cases}$$

iii. Probability By Integration.

$$\begin{aligned}
P(x \leq 50.2) &= \int_{-\infty}^{50.2} f(x)\,dx \\
&= \int_{-\infty}^{49.5} f(x)\,dx + \int_{49.5}^{50.2} f(x)\,dx \\
&= \int_{-\infty}^{49.5} 0\,dx + \int_{49.5}^{50.2} 1\,dx \\
&= 0 + [x]_{49.5}^{50.2} \\
&= 50.2 - 49.5 \;=
\end{aligned}$$

(circle one) **0.5** / **0.7** / **0.9**.

iv. Probability By Distribution Function.

$$\begin{aligned}
P(x \leq 50.2) &= F(50.2) \\
&= \frac{50.2-\alpha}{\beta-\alpha} \\
&= \frac{50.2-49.5}{52-49.5} \;=
\end{aligned}$$

(circle one) **0.5** / **0.7** / **0.9**.

v. $P(x \geq 50.1) = 1 - F(50.2) =$ (circle one) **0.2** / **0.3** / **0.4**.

vi. *Expectation and Variance.*
$E[X] = \frac{\beta+\alpha}{2} = \frac{50.5+49.5}{2} =$ (circle one) **50** / **50.5** / **51**.
$\text{Var}(X) = \frac{(\beta-\alpha)^2}{12} = \frac{(50.5-49.5)^2}{12}$ (circle one) **0.075** / **0.083** / **0.093**.

(c) *Figure (c).*

i. The probability $P([48.4, 52.3]) = P([48.4 \leq x \leq 52.3])$ is represented by or equal to the (circle none, one or more)

A. rectangular area equal to 1.

B. rectangular area equal to the width $(52.3 - 48.4 = 3.9)$ times the height $(\frac{1}{3.9})$.

C. definite integral of $f(x) = \frac{1}{3.9}$ over the interval $[48.4, 52.3]$.

D. $\int_{48.4}^{52.3} \frac{1}{3.9}\,dx = \left[\frac{1}{3.9}x\right]_{48.4}^{52.3} = \frac{1}{3.9}(52.3) - \frac{1}{3.9}(48.4) = 1.$

ii. **True** / **False** The probability density function is given by the piecewise function,

$$f(x) = \begin{cases} \frac{1}{3.9} & \text{if } 48.4 \leq x \leq 52.3 \\ 0 & \text{elsewhere} \end{cases}$$

iii. Probability By Distribution Function.

$$\begin{aligned} P(x \leq 50.2) &= F(50.2) \\ &= \frac{50.2 - \alpha}{\beta - \alpha} \\ &= \frac{50.2 - 48.4}{52.3 - 48.4} = \end{aligned}$$

(circle one) $\frac{\mathbf{1}}{\mathbf{3.9}}$ / $\frac{\mathbf{1.2}}{\mathbf{3.9}}$ / $\frac{\mathbf{2.1}}{\mathbf{3.9}}$.

iv. More Probability By Distribution Function.

$$\begin{aligned} P(49.3 \leq x \leq 50.2) &= F(50.2) - F(47.3) \\ &= \frac{50.2 - 48.4}{52.3 - 48.4} - \frac{49.3 - 48.4}{52.3 - 48.4} = \end{aligned}$$

(circle one) **0.8** / **0.9** / **1.0**.

(d) *More Questions.*

i. If a uniform density is defined on the interval $[40, 50]$, then $f(x) =$ (circle one) $\frac{\mathbf{1}}{\mathbf{10}}$ / $\frac{\mathbf{1}}{\mathbf{15}}$ / $\frac{\mathbf{1}}{\mathbf{20}}$ and zero elsewhere since $\int_{40}^{50} \frac{1}{10}\, dx = 1$

ii. If a uniform density is defined on the interval $[0, 50]$, then $f(x) =$ (circle one) $\frac{\mathbf{1}}{\mathbf{30}}$ / $\frac{\mathbf{1}}{\mathbf{40}}$ / $\frac{\mathbf{1}}{\mathbf{50}}$ and zero elsewhere since $\int_{0}^{50} \frac{1}{50}\, dx = 1$

iii. If a uniform density is defined on the interval $[-10, 50]$, then $f(x) =$ (circle one) $\frac{\mathbf{1}}{\mathbf{30}}$ / $\frac{\mathbf{1}}{\mathbf{40}}$ / $\frac{\mathbf{1}}{\mathbf{60}}$ and zero elsewhere since $\int_{-10}^{50} \frac{1}{60}\, dx = 1$

iv. If a uniform density is defined on the interval $[-10, 50]$, then $P([0, 50]) = \int_{0}^{50} \frac{1}{60}\, dx =$ (circle one) $\frac{\mathbf{30}}{\mathbf{60}}$ / $\frac{\mathbf{40}}{\mathbf{60}}$ / $\frac{\mathbf{50}}{\mathbf{60}}$

v. If a uniform density is defined on the interval $[-2.3, 5.5]$, then $P([-2.1, 5.1]) =$ (circle one) $\frac{\mathbf{7.1}}{\mathbf{7.8}}$ / $\frac{\mathbf{7.2}}{\mathbf{7.8}}$ / $\frac{\mathbf{7.7}}{\mathbf{7.8}}$

vi. **True** / **False**. In general, a uniform probability density function over the interval $[a, b]$ is given by

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{elsewhere} \end{cases}$$

vii. A uniform probability density function has the following properties: (circle none, one or more)

A. continuity (there are *no* “gaps” in the interval).

B. nonnegative (it is never negative).

C. integral over entire interval equals exactly 1.

5.4 Normal Random Variables

We now look at the (standard) normal probability density distribution,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

with distribution function,

$$F(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

One interesting property[1] of the standard normal is

$$1 - \Phi(x) \sim \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

We also look at a more general version of this function called the (nonstandard) normal probability density distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2}$$

with distribution function,

$$F_X(a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

and where the expected value and variance are

$$\begin{aligned} E[X] &= \mu \\ \mathrm{Var}(X) &= \sigma^2 \end{aligned}$$

Exercise 5.4 (Normal Distribution)

1. *Probabilities For Standard Normal: Westville Temperatures.* In Westville, in February, the temperature, x, is assumed to be *standard* normally distributed with mean $\mu = 0^o$ and variance $\sigma^2 = 1^o$.

[1] $a(x) \sim b(x)$ if $\lim_{x\to\infty} \frac{a(x)}{b(x)} = 1$

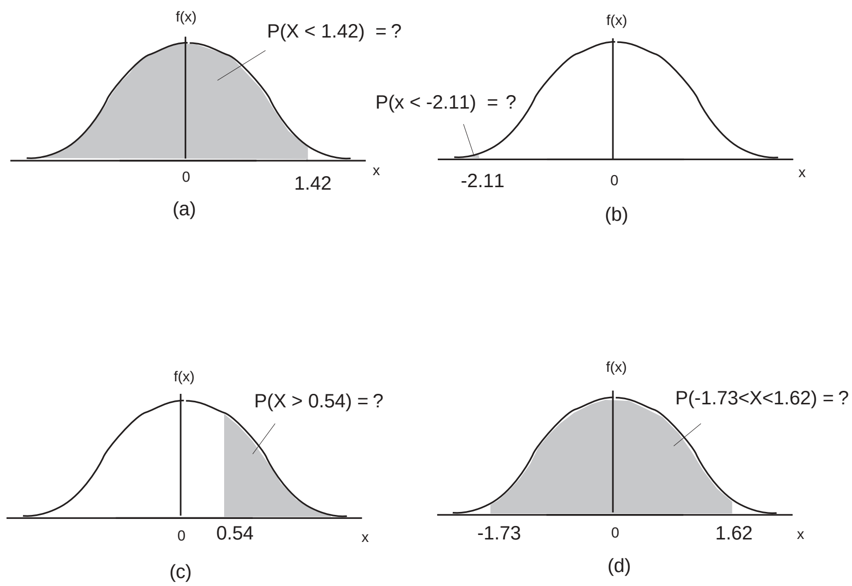


Figure 5.6 (Probabilities For Standard Normal: Westville Temperatures)

(a) The standard normal distribution, in (a) of the figure above, say, is (circle one) **skewed right** / **symmetric** / **skewed left**.

(b) Since the standard normal is a probability density function, the total area under this curve is (circle one) **50%** / **75%** / **100%** / **150%**.

(c) The shape of this distribution is (circle one) **triangular** / **bell–shaped** / **rectangular**.

(d) This distribution has an expected value at (circle one) $\boldsymbol{\mu = 0^o}$ / $\boldsymbol{\mu = 1^o}$.

(e) Since this distribution is symmetric, (circle one) **25%** / **50%** / **75%** of the temperatures are above (to the right) of 0^o.

(f) The probability of the temperature being less than 1.42^o is (circle one) **greater than** / **about the same as** / **smaller than** 0.50. Use (a) in the figure above.

(g) The probability the temperature is less than 1.42^o,

$$\begin{aligned} P\{X \leq 1.42\} &= F(1.42) \\ &= \Phi(1.42) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1.42} e^{-y^2/2}\, dy \ = \end{aligned}$$

0.9222 / **0.0174** / **0.2946** / **0.9056**.
(It is not possible to determine this integral in an analytical way ("by hand") and so you must use your calculator to perform a numerical approximation for this integration: 2nd DISTR 2:normalcdf(− 2nd EE 99, 1.42); look at graph (a) of the figure above to better visualize the probability that is being determined.)

(h) $P\{X < -2.11\} = \Phi(-2.11) =$
(circle one) **0.9222** / **0.0174** / **0.2946** / **0.9056**.
(Use 2nd DISTR 2:normalcdf($-$ 2nd EE 99, -2.11).)

(i) $P\{x > 0.54\} = 1 - \Phi(0.54) =$
(circle one) **0.9222** / **0.0174** / **0.2946** / **0.9056**.
(Use 2nd DISTR 2:normalcdf(0.54, 2nd EE 99).)

(j) $P\{-1.73 < X < 1.62\} = \Phi(1.62) - \Phi(-1.73) =$
(circle one) **0.9222** / **0.0174** / **0.2946** / **0.9056**.
(Use 2nd DISTR 2:normalcdf(-1.73, 1.62).)

(k) **True** / **False** The probability the temperature is *exactly* 1.42^o, say, is *zero*. This is because the probability is equal to the *area* under the bell–shaped curve and there is *no* area "under" the "line" at 1.42^o.

(l) **True** / **False** $P\{X < 1.42^o\} = P\{X \leq 1.42^o\}$.

2. *Nonstandard Normal, A First Look: IQ Scores.* It has been found that IQ scores can be distributed by a nonstandard normal distribution. The following figure compares the two normal distributions for the 16 year olds and 20 year olds.

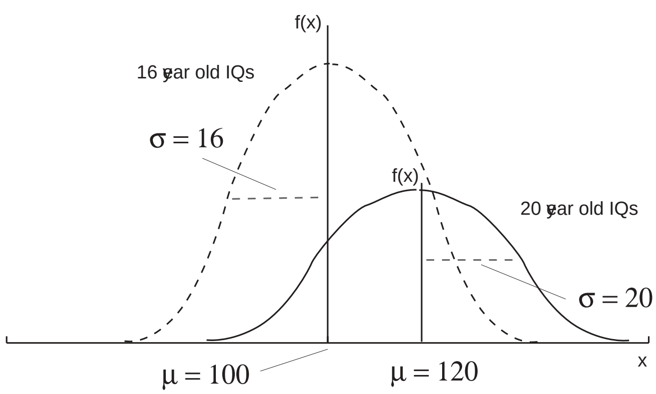


Figure 5.7 (Nonstandard Normal Distributions of IQ Scores)

(a) The mean IQ score for the 20 year olds is
$\mu =$ (circle one) **100** / **120** / **124** / **136**.

(b) The average (or mean) IQ score for the 16 year olds is
(circle one) **100** / **120** / **124** / **136**.

(c) The standard deviation in the IQ score for the 20 year olds
$\sigma =$ (circle one) **16** / **20** / **24** / **36**.

(d) The standard deviation in the IQ score for the 16 year olds is
(circle one) **16** / **20** / **24** / **36**.

(e) The normal distribution for the 20 year old IQ scores is (circle one) **broader than** / **as wide as** / **narrower than** the normal distribution for the 16 year old IQ scores.

(f) The normal distribution for the 20 year old IQ scores is (circle one) **shorter than / as tall as / taller than** than the normal distribution for the 16 year old IQ scores.

(g) The total area (probability) under the normal distribution for the 20 year old IQ scores is (circle one) **smaller than / the same as / larger than** the area under the normal distribution for the 16 year old IQ scores.

(h) **True / False** Neither the normal distribution for the IQ scores for the 20 year old IQ scores nor the 16 year old IQ scores is a *standard* normal because neither have mean zero, $\mu = 0$, and standard deviation 1, $\sigma = 1$. Both, however, have the same general "bell–shaped" distribution.

(i) There is (circle one) **one / two / many / an infinity** of *non*standard normal distributions. The standard normal is one special case of the family of (nonstandard) normal distributions where $\mu = 0$ and $\sigma = 1$.

3. *Probabilities For Nonstandard Normal: IQ Scores Again.*

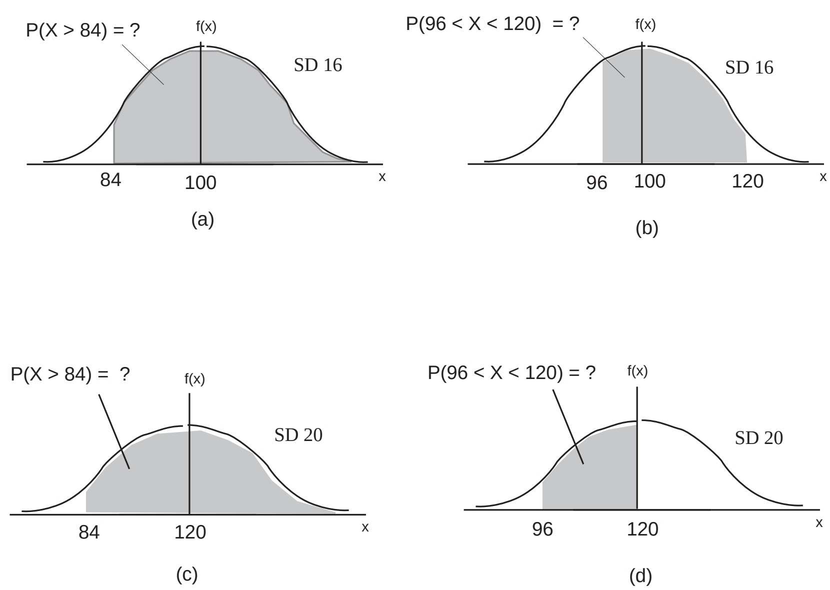


Figure 5.8 (Probabilities For Nonstandard Normal Distributions of IQ Scores)

(a) The *upper two* (of the four) normal curves above represent the IQ scores for *sixteen* year olds. Both are nonstandard normal curves because the (circle none, one or more)

i. the average is 100 and the SD is 16.
ii. neither the average is 0, nor is the SD equal to 1.
iii. the average is 16 and the SD is 100.
iv. the average is 0 and the SD is 1.

The *lower* two normal curves above represent the IQ scores for *twenty* year olds ($\mu = 120, \sigma = 20$).

(b) Since the sixteen year old distribution is symmetric, (circle one) **25%** / **50%** / **75%** of the IQ scores are above (to the right) of 100.

(c) The probability of the IQ scores being less than 84, $P\{X < 84\}$, for the sixteen year old distribution is (circle one) **greater than** / **about the same as** / **smaller than** 0.50.

(d) $P\{X < 84\} =$

$$\begin{aligned} P\{X > 84\} &= 1 - \Phi\left(\frac{84-\mu}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{84-100}{16}\right) \\ &= 1 - \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{84} e^{-(1/2)[(84-100)/16]^2}\, dy \;= \end{aligned}$$

(circle one) **0.8413** / **0.1587** / **−0.1587**
(Use 2nd DISTR 2:normalcdf(− 2nd EE 99, 84, 100, 16).)

(e) Consider the following table of probabilities and possible values of probabilities.

Column I	Column II
(a) $P\{X > 84\}$, "sixteen year old" normal	**(a)** 0.4931
(b) $P\{96 < X < 120\}$, "sixteen year old" normal	**(b)** 0.9641
(c) $P\{X > 84\}$, "twenty year old" normal	**(c)** 0.8413
(d) $P\{96 < X < 120\}$, "twenty year old" normal	**(d)** 0.3849

Using your calculator and the figure above, match the four items in column I with the items in column II.

Column I	(a)	(b)	(c)	(d)
Column II				

(f) **True** / **False** $P\{X < 84\}$ for standard normal equals $P\{X < 84\}$ for the *non*standard normal

4. *Standardizing Nonstandard Normal Random Variables. Non*standard random variable X, with mean μ and standard deviation σ, can be "standardized" into a standard random variable Z using the following formula:

$$Z = \frac{X-\mu}{\sigma}$$

(a) The IQ scores for the 16 year olds are normal with $\mu = 100$ and $\sigma = 16$. The standardized value of the nonstandard IQ score of 110 for the 16 year olds, then, is
$Z = \frac{X-\mu}{\sigma} = \frac{110-100}{16} =$ (circle one) **0.625** / **1.255** / **3.455**
and so $P\{X > 110\} = P\{Z > 0.625\}$.
(Compare 2nd DISTR 2:normalcdf(110, 2nd EE 99, 100, 16) with 2nd DISTR 2:normalcdf(0.625, 2nd EE 99, 0, 1).)

(b) The IQ scores for the 20 year olds are normal with $\mu = 120$ and $\sigma = 20$. The standardized value of the nonstandard IQ score of 110 for the 20 year olds, then, is
$Z = \frac{X-\mu}{\sigma} = \frac{110-120}{20} =$ (circle one) **0.5** / **-0.5** / **0.25**.
and so $P\{X > 110\} = P\{Z > -0.5\}$.
(Compare 2nd DISTR 2:normalcdf(110, 2nd EE 99, 120, 20) with 2nd DISTR 2:normalcdf(-0.5, 2nd EE 99, 0, 1).)

(c) If both a 16 year old and 20 year old score 110 on an IQ test, (check none, one or more)

i. the 16 year old is brighter relative to his age group than the 20 year old is relative to his age group

ii. the z–score is higher for the 16 year old than it is for the 20 year old

iii. the z–score allows us to compare the IQ score for a 16 year old with the IQ score for a 20 year old

(d) If $\mu = 100$ and $\sigma = 16$, then
$P\{X > 130\} = P\left\{Z > \frac{130-100}{16}\right\} =$ (circle one) **0.03** / **0.31**

(e) If $\mu = 120$ and $\sigma = 20$, then
$P\{X > 130\} = P\left\{Z > \frac{130-120}{20}\right\} =$ (circle one) **0.03** / **0.31**

(f) If $\mu = 25$ and $\sigma = 5$, then
$P\{27 < X < 32\} = P\left\{\frac{27-25}{5} < Z < \frac{32-25}{5}\right\} =$
(circle one) **0.03** / **0.26** / **0.31**

5. *Normal Approximation To Binomial.* A lawyer estimates she wins 40% of her cases ($p = 0.4$), and this problem is assumed to obey the conditions of a binomial experiment. If the lawyer presently represents $n = 10$ defendants and X represents the number of wins (of the 10 cases), the functional form of the probability is given by,

$$\binom{10}{i}(0.4)^i(0.6)^{10-i}, \quad i = 0, 1, 2, \ldots 10$$

(a) *Binomial, Exactly.* Various quantities connected to this binomial, including,

i. mean or expected value is given by $\mu = np$ (circle one) **4** / **2.4** / **3.0**.

ii. standard deviation is given by $\sigma = \sqrt{np(1-p)}$ (circle one) **4** / **2.4** / **3.0** / **1.55**.

iii. $P\{X \geq 5\} =$ (circle one) **0.367** / **0.289** / **0.577**.
(Use your calculator; subtract 2nd DISTR A:binomcdf(10, 0.4, 4) from one (1).)

(b) *Normal Approximation.* Consider a graph of the binomial and a normal approximation to this distribution below.

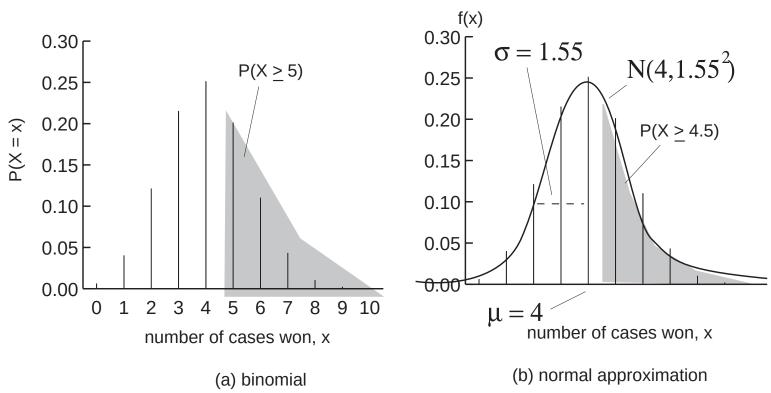


Figure 5.9 (Binomial and Normal Approximation)

i. **True** / **False** The binomial is a discrete distribution, whereas the normal approximation is a continuous distribution. We will be approximating a discrete distribution by a continuous one. In particular, we plan to approximate the shaded $P(X \geq 5)$ from the binomial with the shaded $P(X \geq 4.5)$ from the normal.

ii. It would *appear*, simply by looking at the two graphs above, as though the binomial is (circle one) **skewed** / **symmetric**. This is "good" because we plan to approximate a (not necessarily, but, in this case, apparently) symmetric binomial with an *always* symmetric normal distribution.

iii. To check to see if the binomial is symmetric "enough", we must show that both

$$np \geq 5 \quad \text{and} \quad n(1-p) \geq 5.$$

In fact, these conditions are violated in the following way (circle one)

A. $np \geq 5$ and $n(1-p) \geq 5$

B. $np < 5$ and $n(1-p) \geq 5$

C. $np \geq 5$ and $n(1-p) < 5$

D. $np < 5$ and $n(1-p) < 5$

and so the binomial, in this case, is actually *not* symmetric enough to be approximated by the normal.

iv. However, in spite of violating the conditions required for symmetry, we will proceed to approximate the binomial with a normal. The normal we will use to approximate the binomial with will be a nonstandard normal with mean equal to the mean of the binomial, $\mu = np = 10(4)$, and standard deviation equal to the standard deviation of the binomial, $\sigma = \sqrt{np(1-p)}$ = (circle one) **2.4 / 1.55**.

v. **True / False** The nonstandard normal distribution we will use to approximate the binomial distribution has a mean of 4 and a standard deviation of 1.55.

vi. If X in normal where $\mu = 4$ and $\sigma = 1.55$, then $P\{X \geq 5\}$ = (circle one) **0.374 / 0.259**.
(Use 2nd DISTR 2:normalcdf(5, 2nd EE 99, 4, 1.55).)

vii. The normal approximation, $P\{X \geq 5\} = 0.259$, is (circle one) **smaller than / about the same as / larger than** the exact binomial value, $P\{X \geq 5\} = 0.367$ and so this is a bad normal approximation to the binomial.

viii. To improve the *continuous* normal approximation to the *discrete* binomial, a *continuity correction* factor is introduced. In this case, 0.5 is subtracted from 5 and the revised normal approximation becomes $P\{X \geq 4.5\}$ = (circle one) **0.374 / 0.259**.
(Use 2nd DISTR 2:normalcdf(4.5, 2nd EE 99, 4, 1.55).)

5.5 Exponential Random Variable

We look at the exponential random variable and Laplace (or double exponential) random variable. We also look at the hazard rate (or failure rate) function.

Exercise 5.5 (Exponential Random Variable)

1. *Exponential Random Variable.* The exponential random variable X (often related to the amount of time until a specific event–a telephone call, say–occurs) has a probability density function where

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

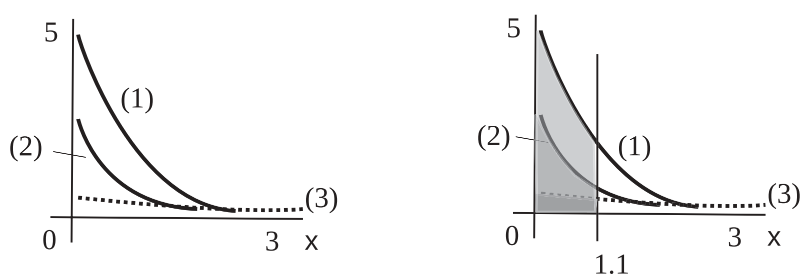
and a distribution function,

$$F(a) = P\{X \leq a\} = 1 - e^{-\lambda a}, \ a \geq 0$$

where the expected value and variance are

$$\begin{aligned} E[X] &= \frac{1}{\lambda} \\ \text{Var}(X) &= \frac{1}{\lambda^2} \end{aligned}$$

(a) *Sketching $\lambda e^{-\lambda x}$: Waiting Time For Emails.* The graphs of different (different λ) exponential density functions are given the figure below.



(a) waiting time densities

(b) chance of waiting less than 1.1 minutes

Figure 5.10 (Exponential Probability Density Functions)

Match each of the exponential distribution functions ((1), (2) and (3)) given below to each of the graphs in (a) given above.

$P =$	$\frac{1}{2}e^{-x/2}$	$3e^{-3x}$	$5e^{-4x}$
graph			

(Hint: Use your calculators; use WINDOW 0 3 1 0 5 1.)

(b) *Figure (a).*

i. For distribution $f(x) = \frac{1}{2}e^{-x/2}$, $\lambda =$ (circle one) $\mathbf{\frac{1}{2}}$ / **3** / **5**

ii. The function $f(x) = \frac{1}{2}e^{-x/2}$ crosses the $f(x)$–axis at $f(x) =$ (circle one) $\mathbf{\frac{1}{2}}$ / **3** / **5**

iii. For distribution $f(x) = 3e^{-3x}$, $\lambda =$ (circle one) $\mathbf{\frac{1}{2}}$ / **3** / **5**

iv. The function $f(x) = 3e^{-3x}$ crosses the $f(x)$–axis at $f(x) =$ (circle one) $\mathbf{\frac{1}{2}}$ / **3** / **5**

v. For distribution $f(x) = 5e^{-5x}$, $\lambda =$ (circle one) $\mathbf{\frac{1}{2}}$ / **3** / **5**

vi. The function $f(x) = 5e^{-5x}$ crosses the $f(x)$–axis at $f(x) =$ (circle one) $\mathbf{\frac{1}{2}}$ / **3** / **5**

vii. The function $f(x) = \frac{1}{2}e^{-x/2}$ is (circle one)
more steeply bent towards the $f(x)$–axis
less steeply bent towards the $f(x)$–axis
as steeply bent towards the $f(x)$–axis
as $f(x) = 5e^{-5x}$.

viii. As positive λ becomes larger, the function $f(x) = \lambda e^{-\lambda x}$,
bends more steeply towards the $f(x)$–axis
bends less steeply towards the $f(x)$–axis
The constant λ is called the *rate* of the distribution.

(c) *Figure (b).*

i. *Probability By Integration.* The probability of waiting less than 1.1 minutes for an email when $\lambda = \frac{1}{2}$ is

$$\begin{aligned} P\{X \leq 1.1\} &= \int_0^{1.1} \lambda e^{-\lambda x}\,dx \\ &= \int_0^{1.1} \frac{1}{2}e^{-\frac{1}{2}x}\,dx = \end{aligned}$$

(circle one) **0.32** / **0.42** / **0.52**.
(Define $\mathrm{Y}_1 = \frac{1}{2}e^{-\frac{1}{2}X}$, then MATH fnInt($\mathrm{Y}_1$,X,0,1.1) ENTER)

ii. Probability By Distribution Function[2].

$$\begin{aligned} P\{X \leq 1.1\} &= F(1.1) \\ &= 1 - e^{-\lambda(1.1)} \\ &= 1 - e^{-\frac{1}{2}(1.1)} = \end{aligned}$$

(circle one) **0.32** / **0.42** / **0.52**.

[2]The distribution function, $F(x)$, is often easier to calculate than integrating the distribution, $\int f(x)\,dx$.

iii. For $\lambda = 3$, $P\{X < 1.1\} = F(1.1) = 1 - e^{-3(1.1)} =$ (circle one) **0.32** / **0.42** / **0.96**.

iv. For $\lambda = 5$, $P\{X < 1.1\} = F(1.1) = 1 - e^{-5(1.1)} =$ (circle one) **0.32** / **0.42** / **0.996**.

v. The probability of waiting less than 1.1 minutes for an email when $\lambda = \frac{1}{2}$ is (circle one) **greater than** / **about the same as** / **smaller than** probability of waiting less than 1.1 minute for an email when $\lambda = 5$.

(d) For $\lambda = 3$,
$P\{X > 0.54\} = 1 - F(0.54) = 1 - \left(1 - e^{-(3)(0.54)}\right) = e^{-(3)(0.54)} =$
(circle one) **0.20** / **0.22** / **0.29**.

(e) For $\lambda = 3$,
$P\{1.13 < X < 1.62\} = F(1.62) - F(1.13) = e^{-(3)(1.13)} - e^{-(3)(1.62)} =$ (circle one) **0.014** / **0.026** / **0.29**.

(f) **True** / **False** The probability the waiting time is *exactly* 1.42 minutes, say, is *zero*.

(g) *Expectation and Variance.*
For $\lambda = \frac{1}{2}$, $E[X] = \frac{1}{\lambda} = \frac{1}{1/2} =$ (circle one) **2** / **3** / **4**.
For $\lambda = 3$, $E[X] = \frac{1}{\lambda} = \frac{1}{3} =$ (circle one) $\mathbf{\frac{1}{2}}$ / $\mathbf{\frac{1}{3}}$ / $\mathbf{\frac{1}{4}}$.
For $\lambda = \frac{1}{2}$, $\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{(1/2)^2}$ (circle one) **2** / **3** / **4**.
For $\lambda = 3$, $\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{3^2}$ (circle one) $\mathbf{\frac{1}{2}}$ / $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{1}{9}}$.

(h) *Memoryless Property: Batteries.* A key property of the exponential random variable is that it is memoryless; that is,

$$P\{X > s + t | X > t\} = P\{X > s\};\ s, t \geq 0$$

The exponential random variable is the *only* random variable to possess this property. Suppose the distribution of the lifetime of batteries, X, is exponential, where $\lambda = 3$.

i. $t = 0$, $s = 10$. Suppose the batteries are new when $t = 0$ and they are 10 hours old when $s = 10$. Then

$$P\{X > 10\} = 1 - F(10) = 1 - (1 - e^{-3(10)}) =$$

(circle one) $\boldsymbol{e^{-10}}$ / $\boldsymbol{e^{-20}}$ / $\boldsymbol{e^{-30}}$

and also

$$\begin{aligned} P\{X > 10 | X > 0\} &= \frac{P\{X > 10, X > 0\}}{P\{X > 0\}} \\ &= \frac{P\{X > 10\}}{P\{X > 0\}} \\ &= \frac{1 - F(10)}{1 - F(0)} \\ &= \frac{1 - (1 - e^{-3(10)})}{1 - (1 - e^{-3(0)})} \\ &= \frac{1 - (1 - e^{-3(10)})}{1} = \end{aligned}$$

(circle one) $\boldsymbol{e^{-10}}$ / $\boldsymbol{e^{-20}}$ / $\boldsymbol{e^{-30}}$
or,

$$P\{X > 10 | X > 0\} = P\{X > 10\}$$

or, in other words the chance a battery lasts at least 10 hours or more, is the same as the chance a battery lasts at least 10 hours more, given that it has already lasted 0 hours or more (which is not too surprising).

ii. $t = 5$, $s = 10$. Suppose the batteries are 5 hours old when $t = 5$ and they are 10 hours old when $s = 10$. Then, once again,

$$P\{X > 10\} = 1 - F(10) = 1 - (1 - e^{-3(10)}) =$$

(circle one) $\boldsymbol{e^{-10}}$ / $\boldsymbol{e^{-20}}$ / $\boldsymbol{e^{-30}}$
and also

$$\begin{aligned} P\{X > 15 | X > 5\} &= \frac{P\{X > 15, X > 5\}}{P\{X > 5\}} \\ &= \frac{P\{X > 15\}}{P\{X > 5\}} \\ &= \frac{1 - F(15)}{1 - F(5)} \\ &= \frac{1 - (1 - e^{-3(15)})}{1 - (1 - e^{-3(5)})} \\ &= \frac{e^{-3(15)}}{e^{-3(5)}} = \end{aligned}$$

(circle one) $\boldsymbol{e^{-10}}$ / $\boldsymbol{e^{-20}}$ / $\boldsymbol{e^{-30}}$
or,

$$P\{X > 15 | X > 5\} = P\{X > 10\}$$

or, in other words, the chance a battery lasts at least 10 hours or more, is the same as the chance a battery lasts at least 15 hours more, given that it has already lasted 5 hours or more. This *is* kind of surprising, because it seems to imply the battery's life starts "fresh" after 5 hours, as though the battery "forgot" about the first five hours of its life.

iii. *What Is Not Being Said.* **True** / **False** Although

$$P\{X > 15|X > 5\} = P\{X > 10\}$$

since $P\{X > 15|X > 5\} \neq P\{X > 15\}$,

$$P\{X > 15\} \neq P\{X > 10\}$$

or, in other words, the (unconditional) chance a battery lasts at least 10 hours or more, is *not* the same as the (unconditional) chance a battery lasts at least 15 hours more.

iv. *What, Then, Is The Memoryless Property of Exponential Distributions?* **True** / **False** The "memoryless" property of the exponential distribution is not so much to do with the notion that random variable X is "forgetting" its previous lifetime in some way, as it is to do with the *shape* of the exponential distribution of X that "falls off" in such a way that the area of a particular *un*conditional probability, given by the ratio of (1)/(2) in the figure below, happens to equal the area of a particular *conditional* probability[3], given by the ratio of (3)/(4).

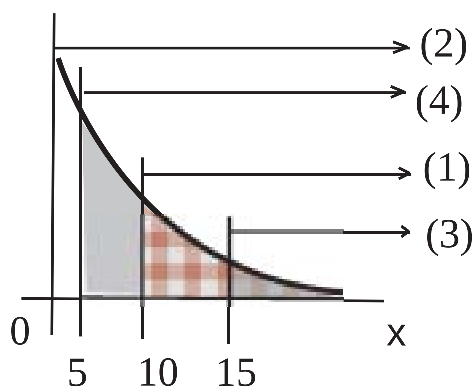


Figure 5.11 (Memoryless Property of Exponential)

v. *An Implication Of The Memoryless Property of Exponential Distributions?* **True** / **False** If

$$P\{X > s + t|X > t\} = P\{X > s\};\ s, t \geq 0$$

[3]In fact, the "rate" of the fall–off of the exponential distribution is traditionally said to have a constant failure (or hazard) rate, as will be discussed shortly.

Review Chapter 6

Jointly Distributed Random Variables

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6.1 Joint Distribution Functions

We look at discrete joint densities in two variables,

$$p(x, y) = P\{X = x, Y = y\}$$

with corresponding marginal densities

$$\begin{aligned} p_X(x) &= P\{X = x\} = \sum_{x:p(x,y)>0} p(x, y) \\ p_Y(y) &= P\{Y = y\} = \sum_{y:p(x,y)>0} p(x, y) \end{aligned}$$

and which has the joint density function in n variables generalization,

$$p(x, y, \ldots, x_n) = P\{X = x, Y = y, \ldots, X_n = x_n\}$$

We also look at continuous joint distribution functions in two variables,

$$F(a, b) = P\{X \leq a, Y \leq b\}, \ -\infty < a, b < \infty$$

with corresponding marginal distributions

$$\begin{aligned} F_X(a) &= P\{X \leq a\} = F(a, \infty) \\ F_Y(b) &= P\{Y \leq b\} = F(\infty, b) \end{aligned}$$

and density[1]

$$f(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b)$$

and which has a joint density function in n variables generalization,

$$F(a_1, a_2, \ldots, a_n) = P\{X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n\}$$

Exercise 6.1 (Joint Distribution Functions)

1. *Discrete Joint Density: Waiting Times To Catch Fish.* The joint density, $P\{X,Y\}$, of the number of minutes waiting to catch the *first* fish, X, *and* the number of minutes waiting to catch the *second* fish, Y, is given below.

	$P\{X=i, Y=j\}$	j 1	2	3	row sum $P\{X=i\}$
	1	0.01	0.02	0.08	0.11
i	2	0.01	0.02	0.08	0.11
	3	0.07	0.08	0.63	0.78
column sum	$P\{Y=j\}$	0.09	0.12	0.79	

   (a) The (joint) chance of waiting *three* minutes to catch the *first* fish and *three* minutes to catch the *second* fish is
   $P\{X=3, Y=3\} =$ (circle one) **0.09** / **0.11** / **0.63** / **0.78**.

   (b) The (joint) chance of waiting *three* minutes to catch the *first* fish and *one* minute to catch the *second* fish is
   $P\{X=3, Y=1\} =$ (circle one) **0.07** / **0.11** / **0.63** / **0.78**.

   (c) The (joint) chance of waiting *one* minute to catch the *first* and *three* minutes to catch the *second* fish is
   $P\{X=1, Y=3\} =$ (circle one) **0.08** / **0.11** / **0.63** / **0.78**.

   (d) The (marginal) chance of waiting *three* minutes to catch the *first* fish is
   $P\{X=3\} =$ (circle one) **0.09** / **0.11** / **0.12** / **0.78**.

   (e) The (marginal) chance of waiting *three* minutes to catch the *second* fish is
   $P\{Y=3\} =$ (circle one) **0.09** / **0.11** / **0.12** / **0.79**.

   (f) The (marginal) chance of waiting *three* minutes to catch the *second* fish is (circle *none, one or more*)

      i. $P\{Y=3\} = 0.79$

[1]Notice that the differentiation is with respect to a and b, rather than X and Y!

ii. $P\{X = 1, Y = 3\} + P\{X = 2, Y = 3\} + P\{X = 3, Y = 3\} = 0.08 + 0.08 + 0.63 = 0.79$
iii. $p_Y(3) = p(1,3) + p(2,3) + p(3,3) = 0.08 + 0.08 + 0.63 = 0.79$
iv. $p_Y(3) = \sum_{y:p(x,y)>0} p(x,y) = p(1,3) + p(2,3) + p(3,3) = 0.08 + 0.08 + 0.63 = 0.79$

(g) The (marginal) chance of waiting *two* minutes to catch the *first* fish is (circle *none, one or more*)

i. $P\{X = 2\} = 0.11$
ii. $P\{X = 2, Y = 1\} + P\{X = 2, Y = 2\} + P\{X = 2, Y = 3\} = 0.01 + 0.02 + 0.08 = 0.11$
iii. $p_X(2) = p(2,1) + p(2,2) + p(2,3) = 0.01 + 0.02 + 0.08 = 0.11$
iv. $p_X(2) = \sum_{y:p(2,y)>0} p(2,y) = p(2,1) + p(2,2) + p(2,3) = 0.01 + 0.02 + 0.08 = 0.11$

(h) The (marginal) chance of waiting *two* minutes to catch the *second* fish is (circle *none, one or more*)

i. $P\{Y = 2\} = 0.12$
ii. $P\{X = 2, Y = 1\} + P\{X = 2, Y = 2\} + P\{X = 2, Y = 3\} = 0.01 + 0.02 + 0.08 = 0.11$
iii. $p_Y(2) = p(1,2) + p(2,2) + p(3,2) = 0.02 + 0.02 + 0.08 = 0.12$
iv. $p_Y(2) = \sum_{y:p(x,2)>0} p(x,2) = p(1,2) + p(2,2) + p(3,2) = 0.02 + 0.02 + 0.08 = 0.12$

(i) The chance of waiting *at least two* minutes to catch the *first* fish is (circle *none, one or more*)

i. $P\{X \geq 2\} = 0.11 + 0.78 = 0.89$
ii. $P\{X = 2, Y = 1\} + P\{X = 2, Y = 2\} + P\{X = 2, Y = 3\} + P\{X = 3, Y = 1\} + P\{X = 3, Y = 2\} + P\{X = 3, Y = 3\} = 0.01 + 0.02 + 0.08 + 0.07 + 0.08 + 0.63 = 0.89$
iii. $p_X(2) = p(2,1) + p(2,2) + p(2,3) = 0.01 + 0.02 + 0.08 = 0.11$
iv. $p_X(2) = \sum_{y:p(2,y)>0} p(2,y) = p(2,1) + p(2,2) + p(2,3) = 0.01 + 0.02 + 0.08 = 0.11$

(j) The chance of waiting *at most two* minutes to catch the *first* fish is (circle *none, one or more*)

i. $P\{X \leq 2\} = 0.11 + 0.11 = 0.22$
ii. $P\{X = 1, Y = 1\} + P\{X = 1, Y = 2\} + P\{X = 1, Y = 3\} + P\{X = 2, Y = 1\} + P\{X = 2, Y = 2\} + P\{X = 2, Y = 3\} = 0.01 + 0.02 + 0.08 + 0.01 + 0.02 + 0.08 = 0.22$
iii. $F(2,3) = P\{X \leq 2, Y \leq 3\} = 0.22$

iv. $F_X(2) = F(2,\infty) = F(2,3) = 0.22$

(k) The chance of waiting *at most two* minutes to catch the *first* fish and *one* minute to catch the *second* fish is (circle *none, one or more*)

i. $P\{X \leq 2, Y = 1\} = 0.11$

ii. $P\{X = 1, Y = 1\} + P\{X = 2, Y = 1\} = 0.01 + 0.02 = 0.03$

iii. $P\{X \leq 2, Y \leq 1\} = F(2,1) = 0.11$

iv. $F_X(2) = F(2,\infty) = F(2,3) = 0.22$

(l) The chance of waiting *at most two* minutes to catch the *first* fish and *at most two* minutes to catch the *second* fish is (circle *none, one or more*)

i. $P\{X \leq 2, Y \leq 2\} = 0.06$

ii. $P\{X = 1, Y = 1\} + P\{X = 2, Y = 2\} + P\{X = 2, Y = 1\} + P\{X = 2, Y = 2\} = 0.01 + 0.02 + 0.01 + 0.02 = 0.06$

iii. $F(2,2) = P\{X \leq 2, Y \leq 2\} = 0.06$

iv. $F_X(2) = F(2,\infty) = F(2,3) = 0.22$

(m) The chance of waiting *at least two* minutes to catch the *first* fish and *at least two* minutes to catch the *second* fish is (circle *none, one or more*)

i. $P\{X \geq 2, Y \geq 2\} = 0.81$

ii. $P\{X > 1, Y > 1\} = 0.81$

iii. $P\{X = 2, Y = 2\} + P\{X = 2, Y = 3\} + P\{X = 3, Y = 2\} + P\{X = 3, Y = 3\} = 0.02 + 0.08 + 0.08 + 0.63 = 0.81$

iv. $1 - F_X(1) - F_Y(1) + F(1,1) = 1 - P\{X \leq 1\} - P\{Y \leq 1\} + P\{X \leq 1, Y \leq 1\} = 1 - 0.11 - 0.09 + 0.01 = 0.81$

Notice that $P\{X \geq 2, Y \geq 2\} \neq 1 - P\{X < 2, Y < 2\}$ because $P\{X \geq 2, Y \geq 2\}$ is the "right–back" portion of the distribution, whereas $P\{X < 2, Y < 2\}$ is the "left–front" portion of the distribution.

2. *Discrete Joint Density: Coin and Dice.* A fair coin, marked "1" on one side and "2" on the other, is flipped once and, independent of this, one fair die is rolled once. Let X be the value of the coin (either 1 or 2) flipped and let Y be the *sum* of the coin flip and die roll (for example, a flip of 2 and a roll of 1 gives $Y = 3$).

(a) The chance of flipping a "1" *and* the sum of coin and die is equal to 2 is

$$\begin{aligned} P\{X = 1, Y = 2\} &= P\{\text{coin is 1, sum is 2}\} \\ &= P\{\text{sum is 2}|\text{coin is 1}\}P\{\text{coin is 1}\} \\ &= P\{\text{die is 1}\}P\{\text{coin is 1}\} \\ &= \frac{1}{6}\cdot\frac{1}{2} \end{aligned}$$

which equals (circle one) $\mathbf{\frac{1}{10}}$ / $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{1}{13}}$.

(b) The chance of flipping a "1" *and* the sum of coin and die is equal to 3 is

$$\begin{aligned} P\{X=1, Y=3\} &= P\{\text{coin is } 1, \text{sum is } 3\} \\ &= P\{\text{sum is } 3 | \text{coin is } 1\} P\{\text{coin is } 1\} \\ &= P\{\text{die is } 2\} P\{\text{coin is } 1\} \\ &= \frac{1}{6} \cdot \frac{1}{2} \end{aligned}$$

which equals (circle one) $\mathbf{\frac{1}{10}}$ / $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{1}{13}}$.

(c) The chance of flipping a "2" *and* the sum of coin and die is equal to 2 is

$$\begin{aligned} P\{X=2, Y=2\} &= P\{\text{coin is } 2, \text{sum is } 2\} \\ &= P\{\text{sum is } 2 | \text{coin is } 2\} P\{\text{coin is } 2\} \\ &= P\{\text{die is } 0\} P\{\text{coin is } 2\} \\ &= 0 \cdot \frac{1}{2} \end{aligned}$$

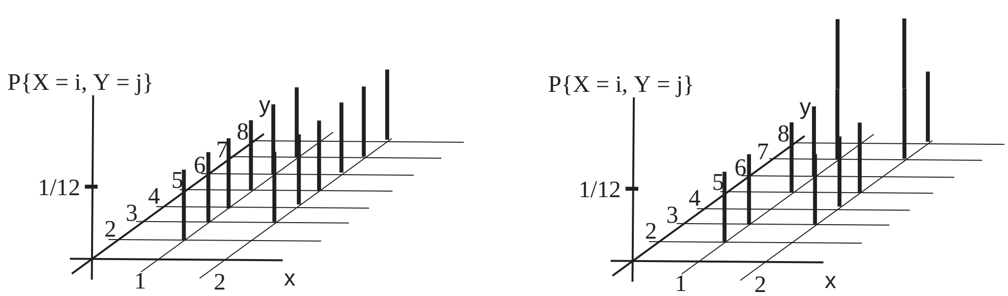
which equals (circle one) **0** / $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{1}{13}}$.

(d) **True** / **False**. The following table is the joint density of $P\{X, Y\}$.

	$P\{X=i, Y=j\}$	j 2	3	4	5	6	7	8	row sum $P\{X=i\}$
	1	1/12	1/12	1/12	1/12	1/12	1/12	0	6/12
i	2	0	1/12	1/12	1/12	1/12	1/12	1/12	6/12
column sum	$P\{Y=j\}$	1/12	2/12	2/12	2/12	2/12	2/12	1/12	

where, notice, the sum of all probabilities is 1.

(e) *Sketching* $P\{X, Y\}$. The possible graphs of $P\{X, Y\}$ are given the figure below.



(a) joint density candidate (b) joint density candidate

Figure 6.1 (Possible $P\{X, Y\}$)

The graph which corresponds to $P\{X, Y\}$, in this case, is graph (circle one) **(a)** / **(b)**.

(f) The chance of flipping a "1" is
$P\{X = 1\} =$ (circle one) $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{2}{12}}$ / $\mathbf{\frac{3}{12}}$ / $\mathbf{\frac{6}{12}}$.

(g) The chance the sum is "8" is
$P\{Y = 8\} =$ (circle one) $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{2}{12}}$ / $\mathbf{\frac{3}{12}}$ / $\mathbf{\frac{6}{12}}$.

(h) The chance the sum is "7" is (circle *none, one or more*)

i. $P\{Y = 7\} = \frac{2}{12}$
ii. $P\{X = 1, Y = 7\} + P\{X = 2, Y = 7\} = \frac{1}{12} + \frac{1}{12} = \frac{2}{12}$
iii. $p_Y(7) = p(1, 7) + p(2, 7) = \frac{1}{12} + \frac{1}{12} = \frac{2}{12}$
iv. $p_Y(7) = \sum_{y:p(x,7)>0} p(x, 7) = \frac{1}{12} + \frac{1}{12} = \frac{2}{12}$

(i) The chance the sum is *at most* "3" is (circle *none, one or more*)

i. $P\{Y \leq 3\} = \frac{3}{12}$
ii. $P\{X = 1, Y = 2\} + P\{X = 1, Y = 3\} + P\{X = 2, Y = 2\} + P\{X = 2, Y = 3\} = \frac{1}{12} + \frac{1}{12} + 0 + \frac{1}{12} = \frac{3}{12}$
iii. $F_Y(3) = F(\infty, 3) = F(2, 3) = \frac{3}{12}$
iv. $F(2, 5) = P\{X \leq 2, Y \leq 5\}$

(j) The chance the coin is *at least* 1 and the sum is *at least* "3" is (circle *none, one or more*)

i. $P\{X \geq 1, Y \geq 3\} = \frac{11}{12}$
ii. $P\{X > 0, Y > 2\} = \frac{11}{12}$
iii. $1 - F_X(0) - F_Y(2) + F(0, 2) = 1 - P\{X \leq 0\} - P\{Y \leq 2\} + P\{X \leq 0, Y \leq 2\} = 1 - 0 - \frac{1}{12} + 0 = \frac{11}{12}$
iv. $P\{X = 2, Y = 2\} + P\{X = 2, Y = 3\} + P\{X = 3, Y = 2\} + P\{X = 3, Y = 3\} = 0.02 + 0.08 + 0.08 + 0.63 = 0.81$

3. *Discrete Joint Density: Marbles In An Urn. Three* marbles are chosen at random with*out* replacement from an urn consisting of 6 black and 8 blue marbles. Let X_i equal 1 if the ith marble selected is black and let it equal 0 otherwise.

(a) *Joint Density of* $P\{X_1, X_2\}$.

i. The chance of, first, choosing a black marble, $X_1 = 1$, and, second, also choosing a black marble, $X_2 = 1$, is
$p(1, 1) =$ (circle one) $\mathbf{\frac{6\cdot 3}{14\cdot 13}}$ / $\mathbf{\frac{6\cdot 4}{14\cdot 13}}$ / $\mathbf{\frac{6\cdot 5}{14\cdot 13}}$ / $\mathbf{\frac{6\cdot 6}{14\cdot 13}}$.
ii. The chance of, first, choosing a black marble, $X_1 = 1$, and, second, choosing a blue marble, $X_2 = 0$, is
$p(1, 0) =$ (circle one) $\mathbf{\frac{6\cdot 6}{14\cdot 13}}$ / $\mathbf{\frac{6\cdot 7}{14\cdot 13}}$ / $\mathbf{\frac{6\cdot 8}{14\cdot 13}}$ / $\mathbf{\frac{6\cdot 9}{14\cdot 13}}$.
iii. **True** / **False** The joint density is

$P\{X_1 = i, X_2 = j\}$		j 0	1	row sum $P\{X_1 = i\}$
i	0	$\frac{8\cdot7}{14\cdot13}$	$\frac{8\cdot6}{14\cdot13}$	$\frac{(6)(8)+(7)(8)}{14\cdot13}$
	1	$\frac{6\cdot8}{14\cdot13}$	$\frac{6\cdot5}{14\cdot13}$	$\frac{(5)(6)+(6)(8)}{14\cdot13}$
column sum	$P\{X_2 = j\}$	$\frac{(6)(8)+(7)(8)}{14\cdot13}$	$\frac{(5)(6)+(6)(8)}{14\cdot13}$	

(b) *Joint Density of* $P\{X_1, X_2, X_3\}$.

i. The chance of, first, choosing a black marble, $X_1 = 1$, and, second, choosing a black marble, $X_2 = 1$, and, third, also choosing a black marble, $X_3 = 1$, is
$p(1,1,1) =$ (circle one) $\frac{\mathbf{6\cdot3\cdot3}}{\mathbf{14\cdot13\cdot12}}$ / $\frac{\mathbf{6\cdot4\cdot3}}{\mathbf{14\cdot13\cdot12}}$ / $\frac{\mathbf{6\cdot5\cdot4}}{\mathbf{14\cdot13\cdot12}}$ / $\frac{\mathbf{6\cdot6\cdot3}}{\mathbf{14\cdot13\cdot12}}$.

ii. The chance of, first, choosing a black marble, $X_1 = 1$, and, second, choosing a black marble, $X_2 = 1$, and, third, choosing a blue marble, $X_3 = 0$, is
$p(1,1,0) =$ (circle one) $\frac{\mathbf{6\cdot3\cdot8}}{\mathbf{14\cdot13\cdot12}}$ / $\frac{\mathbf{6\cdot4\cdot8}}{\mathbf{14\cdot13\cdot12}}$ / $\frac{\mathbf{6\cdot5\cdot8}}{\mathbf{14\cdot13\cdot12}}$ / $\frac{\mathbf{6\cdot6\cdot8}}{\mathbf{14\cdot13\cdot12}}$.

iii. **True** / **False** Since either a black or blue marble can be chosen on each of the three picks out of the urn, there are $2 \times 2 \times 2 = 8$ possible probabilities in the joint density $P\{X_1, X_2, X_3\}$.

4. *Probability Calculations and $Z = XY$: Speeding Tickets.* The joint density, $P\{X, Y\}$, of the number of speeding tickets a driver receives in a year, X, *and* the amount of money required to payoff these tickets, Y, is given below.

$P\{X = i, Y = j\}$		j 20	40	row sum $P\{X = i\}$
i	1	0.2	0.3	0.5
	2	0.4	0.1	0.5
$P\{Y = j\}$ column sum		0.6	0.4	

Determine the density of the total amount of money spent on speeding tickets in a year, $Z = XY$, and use this density to calculate $P\{XY > 20\}$.

(a) The joint distribution probabilities *as well as* the *product*, $Z = XY$, of the number of speeding tickets a driver receives in a year, X, *times* the amount of money required to payoff these tickets, Y, is given below are *combined* in the *one* table below.

$P\{X=i, Y=j\}$ $Z = XY$	j 20	 40	$P\{X=i\}$
i 1	0.2 $1(20) = 20$	0.3 $1(40) = 40$	0.5 4
2	0.4 $2(20) = 40$	0.1 $2(40) = 80$	0.5 5
$P\{Y=j\}$	0.6	0.4	

The *chance* that the total amount paid for speeding tickets in a year is \$20 is given by
(circle one) **0.1** / **0.2** / **0.4** / **0.5**.

(b) The *chance* that the total amount paid for speeding tickets in a year is \$40, $z = xy = 40$, occurs in two possible ways, (2,1) and (1,2), with probabilities (circle one)

i. 0.1 and 0.3, respectively.
ii. 0.2 and 0.3, respectively.
iii. 0.3 and 0.3, respectively.
iv. 0.4 and 0.3, respectively.

(c) Thus, the chance that the total amount paid for speeding tickets in a year is \$40, $z = xy = 40$, is
P$\{XY = 40\} = 0.4 + 0.3 =$ (circle one) **0.4** / **0.6** / **0.7**.

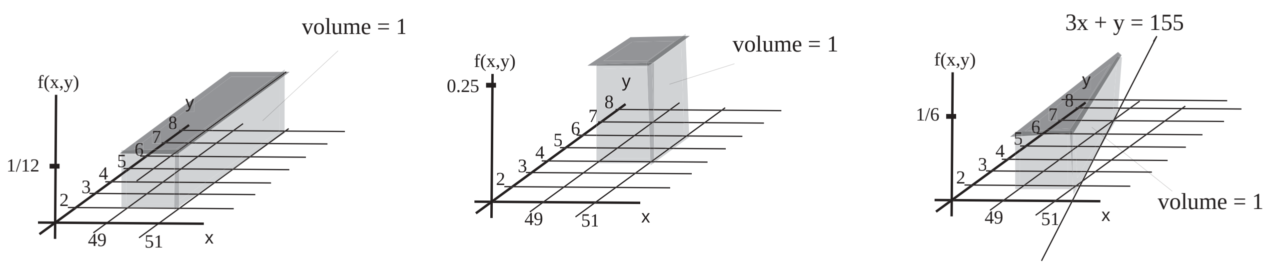
(d) The product, $z = xy = 40$, also occurs in two possible ways (circle one)

i. (2,2) and (1,2).
ii. (1,2) and (2,2).
iii. (1,1) and (2,2).
iv. (2,1) and (1,2).

(e) Complete the probability distribution of the total amount paid for speeding tickets in a year, $Z = XY$,

$z = XY$	20	40	80
$P\{XY\}$	0.2	_____	0.1

5. *Continuous Joint Density: Weight and Amount of Salt in Potato Chips.* Three machines fills potato chip bags. Although each bag should weigh 50 grams each and contain 5 milligrams of salt, in fact, because of differing machines, the weight and amount of salt placed in each bag varies according to the three graphs below.



(a) machine A: f(x,y) = 1/12
49 < x < 51,
2 < y < 8

(b) machine B: f(x,y) = 0.25,
49 < x < 51,
4 < y < 6

(c) machine C: f(x,y) = 1/6,
49 < x < 51, 2 < y < 8,
3x + y < 155

Figure 6.2 (Possible $f(x, y)$)

(a) *Machine A; Figure (a).* One randomly chosen filled bag will weigh *between 49 and 51 grams and* contain *between 2 and 8 milligrams* of salt with probability $P\{49 \leq X \leq 51, 2 \leq Y \leq 8\}$ = (circle one) **1** / **0.5** / **0**.

(b) *More Machine A.* The probability $P\{49 \leq X \leq 51, 2 \leq Y \leq 8\}$ is represented by or equal to the (circle none, one or more)

i. rectangular box volume equal to 1.

ii. rectangular box volume equal to the width $(51 - 49 = 2)$ times the depth $(8 - 2 = 6)$ times the height $(\frac{1}{12})$.

iii. definite integral of $f(x) = \frac{1}{12}$ over the region $(49, 51) \times (2, 8)$.

iv. the integral,

$$\begin{aligned}
\int_2^8 \int_{49}^{51} \frac{1}{12}\, dx\, dy &= \int_2^8 \left[\frac{x}{12}\right]_{49}^{51} dy \\
&= \int_2^8 \left[\frac{51-49}{12}\right] dy \\
&= \left[\frac{2y}{12}\right]_2^8 \\
&= 1
\end{aligned}$$

(c) *More Machine A.* **True** / **False** The *joint probability density function* is given by,

$$f(x, y) = \begin{cases} \frac{1}{12} & 49 \leq x \leq 51, 2 \leq y \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

(d) *More Machine A.* The chance a potato chip bag, chosen at random, weighs *at most* 50.5 grams *and* contains *at most* 4 grams of salt is (circle *none, one or more*)

i. $P\{X \leq 50.5, Y \leq 4\} = (1.5)(2)\frac{1}{12} = \frac{3}{12} = 0.25$

ii. $F(50.5, 4) =$

$$\begin{aligned}
\int_2^4 \int_{49}^{50.5} \frac{1}{12}\, dx\, dy &= \int_2^4 \left[\frac{x}{12}\right]_{49}^{50.5} dy \\
&= \int_2^4 \left[\frac{50.5 - 49}{12}\right] dy \\
&= \left[\frac{1.5y}{12}\right]_2^4 \\
&= \frac{3}{12}
\end{aligned}$$

iii. $P\{X = 1, Y = 1\} + P\{X = 2, Y = 1\} = 0.01 + 0.02 = 0.03$

iv. $P\{X \leq 2, Y \leq 1\} = F(2, 1) = 0.11$

(e) *More Machine A.* The chance a potato chip bag, chosen at random, weighs *at most* 50.5 grams is (circle *none, one or more*)

i. $P\{X \leq 50.5\} = (1.5)(8 - 2)\frac{1}{12} = \frac{9}{12} = 0.75$

ii. $F_X(50.5) = F(50.5, \infty)$

$$\begin{aligned}
\int_2^\infty \int_{49}^{50.5} \frac{1}{12}\, dx\, dy &= \int_2^8 \left[\frac{x}{12}\right]_{49}^{50.5} dy \\
&= \int_2^8 \left[\frac{50.5 - 49}{12}\right] dy \\
&= \left[\frac{1.5x}{12}\right]_2^8 \\
&= \frac{9}{12}
\end{aligned}$$

iii. $F_Y(3) = F(\infty, 3) = F(2, 3) = \frac{3}{12}$

iv. $F(2, 5) = P\{X \leq 2, Y \leq 5\}$

(f) *More Machine A.* The chance a potato chip bag, chosen at random, contains *at most* 4 grams is (circle *none, one or more*)

i. $P\{Y \leq 4\} = (51 - 49)(2)\frac{1}{12} = \frac{4}{12} = 0.33$

ii. $F_Y(4) = F(\infty, 4)$

$$\begin{aligned}
\int_2^4 \int_{49}^\infty \frac{1}{12}\, dx\, dy &= \int_2^4 \left[\frac{x}{12}\right]_{49}^{51} dy \\
&= \int_2^4 \left[\frac{51 - 49}{12}\right] dy \\
&= \left[\frac{2x}{12}\right]_2^4 \\
&= \frac{4}{12}
\end{aligned}$$

iii. $F_Y(3) = F(\infty, 3) = F(2, 3) = \frac{3}{12}$

iv. $F(2, 5) = P\{X \leq 2, Y \leq 5\}$

(g) *More Machine A.* The chance a potato chip bag, chosen at random, weighs *at least* 50.5 grams *and* contains *at least* 4 grams of salt is (circle *none, one or more*)

i. $P\{X \geq 50.5, Y \geq 4\} =$

$$\begin{aligned}
\int_4^8 \int_{50.5}^{51} \frac{1}{12}\, dx\, dy &= \int_4^8 \left[\frac{x}{12}\right]_{50.5}^{51} dy \\
&= \int_4^8 \left[\frac{51 - 50.5}{12}\right] dy \\
&= \left[\frac{0.5x}{12}\right]_4^8 \\
&= \frac{2}{12}
\end{aligned}$$

ii. $P\{X \geq 50.5, Y \geq 4\} =$

$$\begin{aligned}
1 - F_X(50.5) - F_Y(4) + F(50.5, 4) &= 1 - \frac{9}{12} - \frac{4}{12} + \frac{3}{12} \\
&= \frac{2}{12}
\end{aligned}$$

iii. $F_Y(3) = F(\infty, 3) = F(2, 3) = \frac{3}{12}$

iv. $F(2, 5) = P\{X \leq 2, Y \leq 5\}$

Notice that $P\{X \geq 50.5, Y \geq 4\} \neq 1 - P\{X < 50.5, Y < 4\}$ because $P\{X \geq 50.5, Y \geq 4\}$ is the "right–back" portion of the distribution, whereas $P\{X < 50.5, Y < 4\}$ is the "left–front" portion of the distribution.

(h) *Machine B; Figure (b).* One randomly chosen filled bag will weigh *between 49 and 51 grams and* contain *between 4 and 6 milligrams* of salt with probability $P\{49 \leq X \leq 51, 4 \leq Y \leq 6\} =$ (circle one) **1** / **0.5** / **0**.

(i) *More Machine B.* The probability $P\{49 \leq X \leq 51, 4 \leq Y \leq 6\}$ is represented by or equal to the (circle none, one or more)

i. rectangular box volume equal to 1.

ii. rectangular box volume equal to the width $(51 - 49 = 2)$ times the depth $(6 - 4 = 2)$ times the height $(\frac{1}{4})$.

iii. definite integral of $f(x) = \frac{1}{4}$ over the region $(49, 51) \times (4, 6)$.

iv. the integral,

$$\begin{aligned}
\int_4^6 \int_{49}^{51} \frac{1}{4}\,dx\,dy &= \int_4^6 \left[\frac{x}{4}\right]_{49}^{51} dy \\
&= \int_4^6 \left[\frac{51-49}{4}\right] dy \\
&= \left[\frac{2x}{4}\right]_4^6 \\
&= 1
\end{aligned}$$

(j) *More Machine B.* **True** / **False** The *joint probability density function* is given by,

$$f(x,y) = \begin{cases} \frac{1}{4} & 49 \le x \le 51, 4 \le y \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

(k) *More Machine B.* The chance a potato chip bag, chosen at random, weighs *at most* 50.5 grams *and* contains *at most* 5 grams of salt is (circle *none, one or more*)

i. $P\{X \le 50.5, Y \le 5\} = (1.5)(1)\frac{1}{4} = \frac{1.5}{4}$

ii. $F(50.5, 5) =$

$$\begin{aligned}
\int_4^5 \int_{49}^{50.5} \frac{1}{4}\,dx\,dy &= \int_4^5 \left[\frac{x}{4}\right]_{49}^{50.5} dy \\
&= \int_4^5 \left[\frac{50.5-49}{4}\right] dy \\
&= \left[\frac{1.5x}{4}\right]_4^5 \\
&= \frac{1.5}{4}
\end{aligned}$$

iii. $P\{X = 1, Y = 1\} + P\{X = 2, Y = 1\} = 0.01 + 0.02 = 0.03$

iv. $P\{X \le 2, Y \le 1\} = F(2, 1) = 0.11$

(l) *More Machine B.* The chance of potato chip bag, chosen at random, weighs *at most* 50.5 grams is (circle *none, one or more*)

i. $P\{X \le 50.5\} = (1.5)(6-4)\frac{1}{4} = \frac{3}{4} = 0.75$

ii. $F_X(50.5) = F(50.5, \infty)$

$$\begin{aligned}
\int_4^\infty \int_{49}^{50.5} \frac{1}{4}\, dx\, dy &= \int_4^6 \left[\frac{x}{4}\right]_{49}^{50.5} dy \\
&= \int_4^6 \left[\frac{50.5-49}{4}\right] dy \\
&= \left[\frac{1.5y}{4}\right]_4^6 \\
&= \frac{3}{4}
\end{aligned}$$

iii. $F_Y(3) = F(\infty, 3) = F(2, 3) = \frac{3}{12}$

iv. $F(2, 5) = P\{X \le 2, Y \le 5\}$

(m) *More Machine B.* The chance of potato chip bag, chosen at random, contains *at most* 5 grams is (circle *none, one or more*)

i. $P\{Y \le 5\} = (51-49)(1)\frac{1}{4} = \frac{2}{4} = 0.50$

ii. $F_Y(5) = F(\infty, 5)$

$$\begin{aligned}
\int_4^5 \int_{49}^{\infty} \frac{1}{4}\, dx\, dy &= \int_4^5 \left[\frac{x}{4}\right]_{49}^{51} dy \\
&= \int_4^5 \left[\frac{51-49}{12}\right] dy \\
&= \left[\frac{2y}{4}\right]_4^5 \\
&= \frac{2}{4}
\end{aligned}$$

iii. $F_Y(3) = F(\infty, 3) = F(2, 3) = \frac{3}{12}$

iv. $F(2, 5) = P\{X \le 2, Y \le 5\}$

(n) *More Machine B.* The chance of potato chip bag, chosen at random, weighs *at least* 50.5 grams *and* contains *at least* 5 grams of salt is (circle *none, one or more*)

i. $P\{X \ge 50.5, Y \ge 5\} =$

$$\begin{aligned}
\int_5^6 \int_{50.5}^{51} \frac{1}{4}\, dx\, dy &= \int_5^6 \left[\frac{x}{4}\right]_{50.5}^{51} dy \\
&= \int_5^6 \left[\frac{51-50.5}{4}\right] dy \\
&= \left[\frac{0.5y}{4}\right]_5^6 \\
&= \frac{0.5}{4}
\end{aligned}$$

ii. $P\{X \geq 50.5, Y \geq 5\} =$

$$\begin{aligned} 1 - F_X(50.5) - F_Y(5) + F(50.5, 5) &= 1 - \frac{3}{4} - \frac{2}{4} + \frac{1.5}{4} \\ &= \frac{0.5}{4} \end{aligned}$$

iii. $F_Y(3) = F(\infty, 3) = F(2, 3) = \frac{3}{12}$

iv. $F(2, 5) = P\{X \leq 2, Y \leq 5\}$

(o) *Machine C; Figure (c).* One randomly chosen filled bag will weigh, X *between 49 and 51 grams and* contain *between 2 and 8 milligrams* of salt, Y, and also the weight and amount of salt obeys the constraint $3X+Y < 155$, with probability $P\{49 \leq X \leq 51, 2 \leq Y \leq 8\} =$ (circle one) **1** / **0.5** / **0**.

(p) *More Machine C.* The probability $P\{49 \leq X \leq 51.2 \leq Y \leq 8\}$ is represented by or equal to the (circle none, one or more)

i. pie slice volume equal to 1.

ii. pie slice volume equal to the width $(51 - 49 = 2)$ times one–half the depth $(\frac{1}{2}(8-2) = 3)$ times the height $(\frac{1}{6})$.

iii. definite integral of $f(x) = \frac{1}{6}$ over the region $49 < X < 51$, $2 < Y < 8$, $3X + Y < 155$.

iv. the integral,

$$\begin{aligned} \int_2^8 \int_{3x+y<155} \frac{1}{6}\, dx\, dy &= \int_2^8 \int_{x<155/3-(1/3)y} \frac{1}{6}\, dx\, dy \\ &= \int_2^8 \left[\frac{x}{6}\right]_{49}^{155/3-(1/3)y} dy \\ &= \int_2^8 \left[\frac{(155/3-(1/3)y) - 49}{6}\right] dy \\ &= \left[\frac{(8/3)y}{6} - \frac{(1/6)y^2}{6}\right]_2^8 \\ &= 1 \end{aligned}$$

(q) *More Machine C.* **True** / **False** The *joint probability density function* is given by,

$$f(x,y) = \begin{cases} \frac{1}{6} & 49 \leq x \leq 51, 2 \leq y \leq 8, 3x + y < 155 \\ 0 & \text{elsewhere} \end{cases}$$

(r) *More Machine C.* The chance a potato chip bag, chosen at random, weighs *at most* 50.5 grams *and* contains *at most* 4 grams of salt is (circle *none, one or more*)

i. $P\{X \leq 50.5, Y \leq 4\} = (50.5-49)(4-2)\frac{1}{6} - \frac{1}{2}(50.5-151/3)(4-3.5)\frac{1}{6} \approx 0.49305$ because the "back–right pie–shaped corner" of the box volume in $(2,4) \times (49, 50.5)$ is "chopped off".

ii. $F(50.5, 4) =$

$$\begin{aligned}
\int_2^4 \int_{3x+y<155} \frac{1}{6}\, dx\, dy &= \int_2^4 \int_{49}^{50.5} \frac{1}{6}\, dx\, dy - \int_{3.5}^4 \int_{151/3<x<155/3-(1/3)y} \frac{1}{6}\, dx\, dy \\
&= \int_2^4 \left[\frac{x}{6}\right]_{49}^{50.5} dy - \int_{3.5}^4 \left[\frac{x}{6}\right]_{151/3}^{155/3-(1/3)y} dy \\
&= \int_2^4 \left[\frac{50.5-49}{6}\right] dy - \int_{3.5}^4 \left[\frac{(155/3-(1/3)y)-151/3}{6}\right] dy \\
&= \left[\frac{1.5y}{6}\right]_2^4 - \left[\frac{(4/3)y}{6} - \frac{(1/6)y^2}{6}\right]_{3.5}^4 \\
&= 0.5 - 0.006944 \;=\; 0.49305
\end{aligned}$$

iii. $P\{X = 1, Y = 1\} + P\{X = 2, Y = 1\} = 0.01 + 0.02 = 0.03$

iv. $P\{X \leq 2, Y \leq 1\} = F(2,1) = 0.11$

(s) *More Machine C.* The chance a potato chip bag, chosen at random, weighs *at most* 50.5 grams is (circle *none, one or more*)

i. $P\{X \leq 50.5\} = (50.5-49)(8-2)\frac{1}{6} - \frac{1}{2}(50.5-49)(8-3.5)\frac{1}{6} \approx 0.9375$ because the "back–right pie–shaped corner" of the box volume in $(2,8) \times (49, 50.5)$ is "chopped off".

ii. $F_X(50.5) =$

$$\begin{aligned}
\int_2^8 \int_{3x+y<155} \frac{1}{6}\, dx\, dy &= \int_2^8 \int_{49}^{50.5} \frac{1}{6}\, dx\, dy - \int_{3.5}^8 \int_{49<x<155/3-(1/3)y} \frac{1}{6}\, dx\, dy \\
&= \int_2^8 \left[\frac{x}{6}\right]_{49}^{50.5} dy - \int_{3.5}^8 \left[\frac{x}{6}\right]_{49}^{155/3-(1/3)y} dy \\
&= \int_2^8 \left[\frac{50.5-49}{6}\right] dy - \int_{3.5}^8 \left[\frac{(155/3-(1/3)y)-49}{6}\right] dy \\
&= \left[\frac{1.5y}{6}\right]_2^8 - \left[\frac{(8/3)y}{6} - \frac{(1/6)y^2}{6}\right]_{3.5}^8 \\
&= 1.5 - 0.5625 \;=\; 0.9375
\end{aligned}$$

iii. $F_Y(3) = F(\infty, 3) = F(2,3) = \frac{3}{12}$

iv. $F(2,5) = P\{X \leq 2, Y \leq 5\}$

(t) *More Machine C.* The chance a potato chip bag, chosen at random, contains *at most* 4 grams is (circle *none, one or more*)

i. $P\{Y \leq 4\} = (51-49)(4-2)\frac{1}{6} - \frac{1}{2}(51 - 151/3)(4-2)\frac{1}{6} = \frac{5}{9} \approx 0.55$ because the "back–right pie–shaped corner" of the box volume in $(2,4) \times (49,51)$ is "chopped off".

ii. $F_Y(4) =$

$$
\begin{aligned}
\int_2^4 \int_{3x+y<155} \frac{1}{6}\,dx\,dy &= \int_2^4 \int_{49}^{51} \frac{1}{6}\,dx\,dy - \int_2^4 \int_{50.5<x<155/3-(1/3)y} \frac{1}{6}\,dx\,dy \\
&= \int_2^4 \left[\frac{x}{6}\right]_{49}^{51} dy - \int_2^4 \left[\frac{x}{6}\right]_{50.5}^{155/3-(1/3)y} dy \\
&= \int_2^4 \left[\frac{51-49}{6}\right] dy - \int_2^4 \left[\frac{(155/3-(1/3)y)-50.5}{6}\right] dy \\
&= \left[\frac{2y}{6}\right]_2^4 - \left[\frac{(2/3)y}{6} - \frac{(1/6)y^2}{6}\right]_2^4 \\
&= 0.6666 - 0.1111 \;=\; 0.5555
\end{aligned}
$$

iii. $F_Y(3) = F(\infty, 3) = F(2,3) = \frac{3}{12}$

iv. $F(2,5) = P\{X \leq 2, Y \leq 5\}$

(u) *More Machine C.* The chance a potato chip bag, chosen at random, weighs *at least* 50.5 grams *and* contains *at least* 4 grams of salt is (circle *none, one or more*)

i. $P\{X \geq 50.5, Y \geq 4\} = 0$ since the joint density is not defined in this region

ii. $P\{X \geq 50.5, Y \geq 4\} =$

$$
\begin{aligned}
1 - F_X(50.5) - F_Y(4) + F(50.5, 4) &= 1 - 0.9375 - 0.5555 + 0.49305 \\
&= 0
\end{aligned}
$$

iii. $F_Y(3) = F(\infty, 3) = F(2,3) = \frac{3}{12}$

iv. $F(2,5) = P\{X \leq 2, Y \leq 5\}$

6. *More Continuous Distribution Functions.*

(a) If the joint distribution function is given by

$$
F(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}
$$

Then $P\{x \leq 1, y \leq 2\} =$ (circle one)

i. $(1-e^{-(1)^2})(1-e^{-(2)^2}) = 0.621$

ii. $\int_0^2 \int_0^1 (1-e^{-(1)^2})(1-e^{-(2)^2})\,dx\,dy = 0.621$

and the joint density $f(x,y)$ is given by (circle *none, one or more*)

i. $\frac{\partial^2}{\partial x \partial y} F(x,y)$
ii. $\frac{\partial^2}{\partial x \partial y}(1 - e^{-(1)^2})(1 - e^{-(2)^2})$
iii. $[-(-2x)e^{-x^2}] \times [-(2y)e^{-y^2}] = 2xye^{-(x^2+y^2)}$

and $P\{1 < x < 1.25, 1.5 < y < 2\} =$ (circle *none, one or more*)

i. $F(1,1.5) + F(1.25,2) - F(1,2) - F(1.25,1.5)$
ii. $(1 - e^{-(1)^2})(1 - e^{-(1.5)^2}) + (1 - e^{-(1.25)^2})(1 - e^{-(2)^2}) - (1 - e^{-(1)^2})(1 - e^{-(2)^2}) - (1 - e^{-(1.25)^2})(1 - e^{-(1.5)^2})$
iii. $0.56549 + 0.775912 - 0.620542 - 0.707082 = 0.013778$

(Hint: Draw a picture of the rectangular region of integration to convince yourself that adding and subtracting the joint distributions as given above is appropriate.)

(b) Determine c so that

$$f(x,y) = \begin{cases} cx(3x - y) & 0 \leq x \leq 2, 0 \leq y \leq 1, x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

is a joint probability density function. Since

$$\begin{aligned} \int_0^1 \int_0^2 cx(3x - y)\, dx\, dy &= c \int_0^1 \int_0^2 (3x^2 - xy)\, dx\, dy \\ &= c \int_0^1 \left(x^3 - \frac{1}{2}x^2 y \right)_0^2 dy \\ &= c \int_0^1 \left(2^3 - \frac{1}{2}2^2 y \right) dy \\ &= c \left(8y - \frac{1}{4}4y^2 \right)_0^1 \\ &= c \left(8 - \frac{1}{4} \right) \\ &= c \frac{31}{4} \\ &= 1 \end{aligned}$$

and so $c =$ (circle one) $\mathbf{\frac{3}{31}}$ / $\mathbf{\frac{4}{31}}$ / $\mathbf{\frac{5}{31}}$ / $\mathbf{\frac{6}{31}}$.

7. *n–Variable Joint Distributions.*

(a) *Discrete 3–Variable Joint Distribution.* Consider the joint distribution,

$$P\{X = x, Y = y, Z = z\} = \begin{cases} \frac{1}{54}xyz & x = 1,2; y = 1,2,3; z = 1,2 \\ 0 & \text{elsewhere} \end{cases}$$

$P\{X = 1, Y = 2, Z = 2\}$ = (circle *none, one or more*) $\mathbf{\frac{1}{54}(1)(2)(2)}$ / $\mathbf{\frac{4}{54}}$ / $\mathbf{\frac{5}{31}}$ / $\mathbf{\frac{6}{31}}$.
$P\{X = 2, Y = 2, Z = 2\}$ = (circle one) $\mathbf{\frac{1}{54}}$ / $\mathbf{\frac{5}{54}}$ / $\mathbf{\frac{8}{54}}$ / $\mathbf{\frac{12}{54}}$.
$P\{X \leq 2, Y = 2, Z = 2\}$ = (circle one) $\mathbf{\frac{1}{54}}$ / $\mathbf{\frac{5}{54}}$ / $\mathbf{\frac{8}{54}}$ / $\mathbf{\frac{12}{54}}$.

(b) *Discrete Multinomial Joint Distribution.* The *multinomial* joint distribution is given by

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r\} = \frac{n!}{n_1!n_2! \cdots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

where $\sum_{i=1}^r n_i = n$.
Suppose a fair die is rolled 8 times. The chance that 1 appears 3 times, 2 appears once, 3 appears once, 4 appears 3 times, and 5 or 6 does not appear is $P\{X = 1, Y = 2, Z = 2\}$ = (circle *none, one or more*)

i. $P\{X_1 = 3, X_2 = 1, X_3 = 1, X_4 = 3, X_5 = 0, X_6 = 0\}$
ii. $\frac{8!}{3!1!1!3!0!0!}(1/6)^3(1/6)^1(1/6)^1(1/6)^3(1/6)^1(1/6)^0$
iii. $\frac{8!}{3!1!1!3!0!0!}(1/6)^8$
iv. 0.0006668

6.2 Independent Random Variables

Random variables X and Y are independent if and only if any of the following equations are satisfied

$$\begin{aligned}
P\{X \in A, Y \in B\} &= P\{X \in A\}P\{Y \in B\} \\
P\{X \leq a, Y \leq b\} &= P\{X \leq a\}P\{X \leq b\} \\
F(a,b) &= F_X(a)F_Y(b) \\
p(x,y) &= p_X(a)p_Y(b), \text{ for all} x, y; \text{(discrete case)} \\
f(x,y) &= f_X(x)f_Y(y), \text{ (continuous case)}
\end{aligned}$$

In general, n random variables X_1, X_2, ... , X_n are independent if, for all sets of real numbers,

$$\begin{aligned}
P\{X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\} &= \Pi_{i=1}^n P\{X_i \in A_i\} \\
P\{X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n\} &= \Pi_{i=1}^n P\{X_i \leq a_i\}
\end{aligned}$$

Exercise 6.2 (Joint Distribution Functions)

1. *Discrete Distribution and Independence: Waiting Time To Fish.* The joint density of the number of minutes waiting to catch a fish on the first and second day, $P\{X, Y\}$, is given below.

$P\{X,Y\}$		Y 1	2	3	row sum $P\{X=x\}$
X	1	0.01	0.02	0.08	0.11
	2	0.01	0.02	0.08	0.11
	3	0.07	0.08	0.63	0.78
column sum	$P\{Y=y\}$	0.09	0.12	0.79	

Are the waiting times on the two days independent of one another? To demonstrate independence, we must show that $P\{X,Y\} = P\{X\}P\{Y\}$ for $X, Y = 1, 2, 3$.

(a) $X = 3$, $Y = 3$.
The chance of waiting *three* minutes to catch one fish on the *first* day is $P\{X = 3\} =$ (circle one) **0.09** / **0.11** / **0.12** / **0.78**.
The chance of waiting *three* minutes to catch one fish on the *second* day is $P\{Y = 3\} =$ (circle one) **0.09** / **0.11** / **0.12** / **0.79**.
The chance of waiting *three* minutes to catch one fish on the *first* day *and* waiting three minutes to catch one fish on the *second* day is $P\{X = 3, Y = 3\} =$ (circle one) **0.09** / **0.11** / **0.63** / **0.78**.
Since the chance of waiting three minutes to catch one fish on the *first* day *and* waiting three minutes to catch one fish on the *second* day,
$P\{X = 3, Y = 3\} = 0.63$,
(circle one) **does** / **does not** equal
$P\{X = 3\}P\{Y = 3\} = (0.78)(0.79) = 0.6162$,
the waiting three minutes on the second day *depends* on the waiting three minutes on the first day.

(b) $X = 2$, $Y = 3$.
The chance of waiting *two* minutes to catch a fish on the *first* day is $P\{X = 2\} =$ (circle one) **0.09** / **0.11** / **0.12** / **0.78**.
The chance of waiting *three* minutes to catch a fish on the *second* day is $P\{Y = 3\} =$ (circle one) **0.09** / **0.11** / **0.12** / **0.79**.
The chance of waiting *two* minutes to catch a fish on the *first* day *and* waiting three minutes to catch a fish on the *second* day is $P\{X = 3, Y = 3\} =$ (circle one) **0.08** / **0.11** / **0.63** / **0.78**.
Since the chance of waiting two minutes to catch a fish on the *first* day *and* waiting three minutes to catch a fish on the *second* day,
$P\{X = 2, Y = 3\} = 0.08$,
(circle one) **does** / **does not** equal
$P\{X = 2\}P\{Y = 3\} = (0.11)(0.79) = 0.0869$,
the waiting three minutes on the second day *depends* on the waiting two minutes on the first day.

(c) **True** / **False** In order for the waiting time on the second day to be independent of the waiting time on the first day, it must be shown that the waiting times of one, two or three minutes on the second day must *all* be shown to be independent of the waiting times of one, two or three minutes on the first day. If *any* of the waiting times on the second day are shown to be independent of any of the waiting times on the first day, this would demonstrate the waiting time on the second day depends on the waiting time of the first day.

2. *More Discrete Distribution and Independence: Waiting Time To Fish Again.* The joint density of the number of minutes waiting to catch a fish on the first and second day, $P\{X, Y\}$, is given below.

	$P\{X, Y\}$	Y 1	2	3	row sum $P\{X = x\}$
	1	0.01	0.01	0.08	0.10
X	2	0.01	0.01	0.08	0.10
	3	0.08	0.08	0.64	0.80
column sum	$P\{Y = y\}$	0.10	0.10	0.80	

Are the waiting times on the two days independent of one another? To demonstrate independence, we must show that $P\{X, Y\} = P\{X\}P\{Y\}$ for $X, Y = 1, 2, 3$.

(a) $X = 2$, $Y = 3$.
The chance of waiting *two* minutes to catch a fish on the *first* day is
$P\{X = 2\} =$ (circle one) **0.09** / **0.10** / **0.12** / **0.78**.
The chance of waiting *three* minutes to catch a fish on the *second* day is
$P\{Y = 3\} =$ (circle one) **0.09** / **0.11** / **0.12** / **0.80**.
The chance of waiting *two* minutes to catch a fish on the *first* day *and* waiting three minutes to catch a fish on the *second* day is
$P\{X = 3, Y = 3\} =$ (circle one) **0.08** / **0.11** / **0.64** / **0.80**.
Since the chance of waiting two minutes to catch a fish on the *first* day *and* waiting three minutes to catch a fish on the *second* day,
$P\{X = 2, Y = 3\} = 0.08$,
(circle one) **does** / **does not** equal
$P\{X = 2\}P\{Y = 3\} = (0.10)(0.80) = 0.08$,
the waiting three minutes on the second day is
(circle one) **independent** / **dependent**
on the waiting two minutes on the first day.

(b) **True** / **False** In fact, since $P\{X, Y\} = P\{X\}P\{Y\}$ for $X, Y = 1, 2, 3$, the waiting time on the second day is *in*dependent of the waiting time on the first day.

3. *More Discrete Joint Distributions and Independence: Waiting Times To Fish Yet Again.* The distribution of the number of minutes waiting to catch a fish, X, on any day is given below.

X	1	2	3
$P\{X = x\}$	0.1	0.1	0.8

   (a) The chance of waiting *one* minute to catch a fish is
   $P\{X = 1\}$ = (circle one) **0.1** / **0.4** / **0.5** / **0.8**.

   (b) The chance of waiting *three* minutes to catch another fish is
   $P\{X = 3\}$ = (circle one) **0.1** / **0.4** / **0.5** / **0.8**.

   (c) Since waiting one minute *and* waiting three minutes are *in*dependent of one another,
   $P\{X = 1, X = 3\} = P\{X = 1\}P\{X = 3\} = (0.1)(0.8) =$
   (circle one) **0.01** / **0.04** / **0.05** / **0.08**.

   (d) Complete the following joint distribution table, if the waiting times on one day are *independent* of the waiting times on any other day.

$P\{X, Y\}$		Y		
		1	2	3
	1	0.01	0.01	0.08
X	2	______	0.01	______
	3	0.08	______	______

4. *More Discrete Joint Distributions and Independence: Shooting Hoops.* Suppose two basketball players are each taking a free throw. Basketball player A has a 45% chance of making a free throw, and so the chance s/he makes a basket on the fourth throw is (using the geometric distribution)

$$P\{X = 4\} = p(1 - p)^3 = 0.45(1 - 0.45)^3$$

Basketball player B also has a 45% chance of making a free throw, and so the chance s/he makes a *second* basket on the fourth throw is (using the negative binomial distribution)

$$P\{Y = 4\} = \binom{i-1}{r-1} p^r (1 - p)^{i-r} = \binom{4-1}{2-1} 0.45^2 (1 - 0.45)^{4-2}$$

Assume basketball player A's free throws are *in*dependent of basketball player B's free throws,

   (a) The chance that A makes a basket on the fourth throw and B makes a second basket on the fourth throw is

$P\{X = 4, Y = 4\} = P\{X = 4\}P\{Y = 4\} =$ (circle one)
$\mathbf{0.45(1-0.45)^1 \times \binom{4-1}{2-1} 0.45^2(1-0.45)^{4-2}}$
$\mathbf{0.45(1-0.45)^2 \times \binom{4-1}{2-1} 0.45^2(1-0.45)^{4-2}}$
$\mathbf{0.45(1-0.45)^3 \times \binom{4-1}{2-1} 0.45^2(1-0.45)^{4-2}}$

(b) The chance that A makes a basket on the third throw and B makes a second basket on the fifth throw is
$P\{X = 3, Y = 5\} = P\{X = 3\}P\{Y = 5\} =$ (circle one)
$\mathbf{0.45(1-0.45)^1 \times \binom{5-1}{2-1} 0.45^2(1-0.45)^{5-2}}$
$\mathbf{0.45(1-0.45)^2 \times \binom{5-1}{2-1} 0.45^2(1-0.45)^{5-2}}$
$\mathbf{0.45(1-0.45)^3 \times \binom{5-1}{2-1} 0.45^2(1-0.45)^{5-2}}$

(c) Suppose A makes a basket on the third throw, starts again and, independent of the first round of throws, makes a basket on the fifth throw on the second round of throws and then, independent of this, on a third round of throws, makes a basket on the first attempt. The chance of this happening is
$P\{X_1 = 3, X_2 = 5, X_3 = 1\} = P\{X_1 = 3\}P\{X_2 = 5\}P\{X_3 = 1\} =$
(circle one)
$\mathbf{0.45(1-0.45)^2 \times 0.45(1-0.45)^2 \times 0.45}$
$\mathbf{0.45(1-0.45)^2 \times 0.45(1-0.45)^3 \times 0.45}$
$\mathbf{0.45(1-0.45)^2 \times 0.45(1-0.45)^4 \times 0.45}$

5. *Continuous Distribution and Independence.* Consider the joint density,

$$f(x,y) = \begin{cases} 4xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

To demonstrate independence of X and Y, we must show that $f(x,y) = f_X(x)f_Y(y)$.

(a) $f_X(x) =$

$$\begin{aligned} \int_0^1 4xy\,dy &= \left(2xy^2\right)_{y=0}^1 \\ &= \left(2x(1)^2 - 2x(0)^2\right) = \end{aligned}$$

(circle one) $\mathbf{2}$ / $\boldsymbol{2x}$ / $\boldsymbol{2y}$ / $\boldsymbol{4xy}$.

(b) $f_Y(y) =$

$$\begin{aligned} \int_0^1 4xy\,dx &= \left(2x^2y\right)_{x=0}^1 \\ &= \left(2(1)^2y - 2(0)^2y\right) = \end{aligned}$$

(circle one) **2** / **2*x*** / **2*y*** / **4*xy***.

(c) Since $f(x, y) = 4xy$
(circle one) **does** / **does not** equal
$f_X(x)f_Y(y) = (2x)(2y) = 4xy$,
random variable X is *in*dependent of Y.

(d) Is independence *symmetric*? If X is independent of Y, then Y (circle one) **is** / **is not** independent of X.

6. *Another Continuous Joint Distribution and Independence.* Consider the joint density,

$$f(x, y) = \begin{cases} 24xy & 0 < x < 1, 0 < y < 1, 0 < x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Are X and Y independent?

(a) $f_X(x) =$

$$\begin{aligned} \int_{x+y<1} 24xy\,dy &= \int_{y<1-x} 24xy\,dy \\ &= \int_0^{1-x} 24xy\,dy \\ &= \left(12xy^2\right)_{y=0}^{1-x} \\ &= \left(12x(1-x)^2 - 12x(0)^2\right) = \end{aligned}$$

(circle one) **2** / **2*x*** / **2*y*** / $\boldsymbol{12x(1-x)^2}$.

(b) $f_Y(y) =$

$$\begin{aligned} \int_{x+y<1} 24xy\,dx &= \int_{x<1-y} 24xy\,dx \\ &= \int_0^{1-y} 24xy\,dx \\ &= \left(12x^2y\right)_{x=0}^{1-y} \\ &= \left(12(1-y)^2y - 12(0)^2y\right) = \end{aligned}$$

(circle one) **2** / **2*x*** / **2*y*** / $\boldsymbol{12y(1-y)^2}$.

(c) Since $f(x,y) = 24xy$
(circle one) **does** / **does not** equal
$f_X(x)f_Y(y) = (12x(1-x)^2)(12y(1-y)^2)$,
random variable X is *dependent* on Y.

7. *Another Continuous Joint Distribution and Independence.* The joint distribution function is given by

$$F(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

To demonstrate independence of X and Y, we must show that $F(x,y) = F_X(x)F_Y(y)$.

(a) $F_X(x) =$

$$\begin{aligned} F(x,\infty) &= \lim_{y\to\infty} F(x,y) \\ &= \lim_{y\to\infty} (1-e^{-x^2})(1-e^{-y^2}) \\ &= (1-e^{-x^2})(1-0) \ = \end{aligned}$$

(circle one) $\mathbf{2}$ / $\boldsymbol{2x}$ / $\boldsymbol{2y}$ / $\boldsymbol{1-e^{-x^2}}$.

(b) $F_Y(y) =$

$$\begin{aligned} F(\infty,y) &= \lim_{x\to\infty} F(x,y) \\ &= \lim_{x\to\infty} (1-e^{-x^2})(1-e^{-y^2}) \\ &= (1-0)(1-e^{-y^2}) \ = \end{aligned}$$

(circle one) $\mathbf{2}$ / $\boldsymbol{2x}$ / $\boldsymbol{2y}$ / $\boldsymbol{1-e^{-y^2}}$.

(c) Since $f(x,y) = (1-e^{-x^2})(1-e^{-y^2})$
(circle one) **does** / **does not** equal
$f_X(x)f_Y(y) = (1-e^{-x^2})(1-e^{-y^2})$,
random variable X is *in*dependent of Y.

8. *Another Continuous Joint Distribution and Independence.* If X and Y are independent, what is the density of X/Y, if X and Y are both exponential random variables with parameters λ and μ, respectively?

(a) *General Density.*

$$\begin{aligned} F_Z(a) &= P\{X/Y < a\} \\ &= P\{X < aY\} \\ &= \int_0^\infty \int_0^{ay} f_X(x)f_Y(y)\,dx\,dy \\ &= \int_0^\infty F_X(ay)f_Y(y)\,dy \end{aligned}$$

and so

$$f_Z(a) = \frac{d}{da} \int_0^\infty F_X(ay) f_Y(y)\, dy$$

(circle one)
$\boldsymbol{\int_0^\infty f_X(ay) f_Y(y)\, dy}$
$\boldsymbol{\int_0^\infty f_X(ay) y f_Y(y)\, dy}$
$\boldsymbol{-\int_0^\infty f_X(ay) y f_Y(y)\, dy}$.

(b) *Density When X and Y Are Exponential Random Variables.* If X is exponential with parameter λ and Y is exponential with parameter μ, $f_Z(a)$ (circle one)
$\boldsymbol{\int_0^\infty \lambda e^{-\lambda ay} \mu e^{-\mu y}\, dy}$
$\boldsymbol{\int_0^\infty \lambda e^{-\lambda ay} y \mu e^{-\mu y}\, dy}$
$\boldsymbol{-\int_0^\infty \lambda e^{-\lambda ay} y \mu e^{-\mu y}\, dy}$.

6.3 Sums of Independent Random Variables

In the discrete case, if X and Y are independent, the distribution of $X + Y$ is[2]

$$\begin{aligned} P\{X + Y = n\} &= P\{X = k, Y = n - k\} \\ &= P\{X = k\} P\{Y = n - k\} \end{aligned}$$

In the continuous case, if X and Y are independent the distribution of $X + Y$ is[3]

$$\begin{aligned} F_{X+Y}(a) &= P\{X + Y \leq a\} \\ &= P\{X \leq a - Y\} \\ &= \int_{-\infty}^{\infty} \int_0^{a+y} f_{X,Y}(x, y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} \int_0^{a-y} f_X(x) f_Y(y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\, dy \end{aligned}$$

and so

$$f_{X+Y}(a) = \frac{d}{da} \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\, dy = \int_{-\infty}^{\infty} f_X(a - y) f_Y(y)\, dy$$

Exercise 6.3 (Statistic and its Sampling Distribution)

[2]Notice the neat trick in equating $P\{X + Y = n\}$ with $P\{X = k, Y = n - k\}$: if $X = k$, $k \leq n$, then $X + Y = k + Y = n$ or $Y = n - k$.

[3]Notice that the trick in equating $P\{X + Y = n\}$ with $P\{X = k, Y = n - k\}$ in the discrete case is effectively repeated here in the continuous case, where $f_{X,Y}(x, y)$ is equated with $f_X(a - y) f_Y(y)$.

1. *Discrete Probability Distribution of Sum, Montana Fishing Trip.* A fisherman takes *two* trips to a lake there, where the number of fish caught at a lake on either one of these two trips, X, is a random variable with the following distribution,

x	1	2	3
$P\{X = x\}$	0.4	0.4	0.2

(a) One fish is caught on the first trip and three fish are caught on the second trip. The *sum* of the number of fish caught over these two trips is $x1 + x_2 = 1 + 3 =$ (circle one) **0.3** / **1.5** / **4**.

(b) Two fish are caught on the first trip and three fish are caught on the second trip. The *sum* of the number of fish caught over these two trips is $x_1 + x_2 = 2 + 3 =$ (circle one) **0.3** / **1.5** / **5**.

(c) The joint distribution probabilities *as well as* the sum of the number of fish caught on two trips to the lake are *combined* in the *one* table below.

	$\mathrm{P}\{x_1, x_2\}$ $x_1 + x_2$	1	x_2 2	3
	1	0.16 2	0.16 3	0.08 4
x_1	2	0.16 3	0.16 4	0.08 5
	3	0.08 4	0.08 5	0.04 6

The sum for when three fish are caught on the first trip and two fish are caught on the second trip, $(3, 2)$, is 5 with chance given by (circle one) **0.04** / **0.08** / **0.16**.

(d) The sum, $x_1 + x_2 = 4$, occurs in three possible ways: (3,1), (2,2) and (1,3), with probabilities (circle one)

i. 0.08, 0.08 and 0.08, respectively.
ii. 0.08, 0.16 and 0.16, respectively.
iii. 0.08, 0.16 and 0.08, respectively.
iv. 0.16, 0.16 and 0.08, respectively.

(e) Thus, the chance that the sum of the number of fish caught on two trips to the lake is *four* is $\mathrm{P}\{X_1 + X_2 = 4\} = 0.08 + 0.16 + 0.08 =$ (circle one) **0.04** / **0.16** / **0.32**.

(f) The sum, $x_1 + x_2 = 5$, occurs in two possible ways (circle one)

i. (2,2) and (1,3).
ii. (2,3) and (2,3).

iii. (3,2) and (2,3).
iv. (2,1) and (1,3).

(g) Combining the probabilities associated with the two ways that the sum 5 can occur,
P($X_1X_2 = 5$) = (circle one) **0.04** / **0.16** / **0.32**.

(h) Complete the *probability distribution* of the sum of the number of fish on two trips to the lake, $X_1 + X_2$,

$x_1 + x_2$	2	3	4	5	6
$P\{X_1 + X_2\}$	0.16	____	0.32	0.16	0.04

(i) **True** / **False** Since the number of fish on each trip to the lake, X, is a random variable, the sum of the number of fish caught, $X_1 + X_2$, is also a random variable.

(j) **True** / **False**. In addition to the probability distribution for the sum, $X_1 + X_2$, there is also a probability distribution for other functions of X_1 and X_2, such as X_1X_2, say.

2. *Discrete Probability Distribution of Sum, Waiting Time To Catch a Fish.* The distribution of the number of minutes waiting to catch a fish, Y, is given below.

x	1	2	3
$P\{X = x\}$	0.1	0.1	0.8

(a) If two minutes are spent waiting for one fish and two minutes are spent waiting for another fish, (2,2), the *sum* of time spent waiting is
$x_1 + x_2 = 2 + 2 =$ (circle one) **0.3** / **1.5** / **4**.

(b) Complete the following table of joint distribution probabilities *as well as* the average times spent waiting to catch two fish.

	$P\{x_1, x_2\}$ $X_1 + x_2$	1	x_1 2	3
	1	0.01 2	0.01 3	0.08 4
x_2	2	0.01 ____	0.01 4	0.08 5
	3	0.08 ____	0.08 5	0.64 6

(c) Complete the probability distribution of the sum of the waiting time, $X_1 + X_2$, is

$x_1 + x_2$	2	3	4	5	6
$P\{X_1 + X_2\}$	____	0.02	0.17	____	0.64

(d) The method used here of determining the probability distribution of $X_1 + X_2$ (circle one) **does** / **does not** require that the random variables X_1 and X_2 are independent of one another; this method could also be applied to random variables that are *dependent*.

3. *Discrete: Poisson sum.* The Poisson random variable has distribution given by

$$p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!},\ i = 0, 1, \ldots, \lambda > 0$$

with parameter λ and so the distribution of the sum $X + Y$ is

$$\begin{aligned}
P\{X+Y=n\} &= \sum_{k=0}^{n} P\{X=k, Y=n-k\} \\
&= \sum_{k=0}^{n} P\{X=k\}P\{Y=n-k\} \\
&= \sum_{k=0}^{n} e^{-\lambda_1}\frac{{\lambda_1}^k}{k!}e^{-\lambda_2}\frac{{\lambda_2}^{n-k}}{(n-k)!} \\
&= e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{{\lambda_1}^k{\lambda_2}^{n-k}}{k!(n-k)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}{\lambda_1}^k{\lambda_2}^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n \text{ binomial coefficient identity}
\end{aligned}$$

(a) Suppose the number of people who live to 100 years of age in Westville per year has a Poisson distribution with parameter $\lambda_1 = 2.5$; whereas, independent of this, in Michigan City, it has a Poisson distribution with parameter $\lambda_2 = 3$. The chance that the *sum* of the number of people who live to 100 years of age in Westville and Michigan City is 4 is
$P\{X_1 + X_2 = 4\} =$ (circle *none, one or more*)
   i. $\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n = \frac{e^{-(2.5+3)}}{4!}(2.5+3)^4$
   ii. 0.1558

   (Hint: Poissonpdf(5.5,4))

(b) For $\lambda_1 = 2.5$, $\lambda_2 = 3$, $P\{X_1 + X_2 \leq 4\} =$
(circle one) **0.311** / **0.358** / **0.543**.
(Hint: Poissoncdf(5.5,4))

(c) For $\lambda_1 = 2.5$, $\lambda_2 = 6$, $P\{X_1 + X_2 \leq 4\} =$
(circle one) **0.074** / **0.358** / **0.543**.

4. *Continuous Example: gamma.* The gamma random variable, with parameters (t, λ), $t \geq 0$, $\lambda \geq 0$ is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{t-1}}{\Gamma(t)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and so the distribution of the sum $X + Y$, where Y has parameters (s, λ), is

$$\begin{aligned} f_{X+Y}(a) &= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy \\ &= \frac{1}{\Gamma(s)\Gamma(t)} \int_0^a \lambda e^{-\lambda(a-y)}(\lambda(a-y))^{s-1} \lambda e^{-\lambda y}(\lambda y)^{t-1}\, dy \\ &= K e^{-\lambda a} \int_0^a (a-y)^{s-1} y^{t-1}\, dy \\ &= K e^{-\lambda a} a^{s+t-1} \int_0^1 (1-x)^{s-1} x^{t-1}\, dx \quad \text{letting } x = \frac{y}{a} \\ &= C e^{-\lambda a} a^{s+t-1} \\ &= \frac{\lambda e^{-\lambda a}(\lambda a)^{s+t-1}}{\Gamma(s+t)} \quad \text{letting } C = \frac{1}{\Gamma(s+t)} \end{aligned}$$

In general, if X_i is gamma with parameter (t_i, λ), then $\sum_{i=1}^{n} X_i$ is gamma with parameter $(\sum_{i=1}^{n} t_1, \lambda)$.

(a) Suppose the number of people who live to 100 years of age in Westville per year has a gamma distribution with parameter $(t, \lambda) = (2, 2.5)$; whereas, independent of this, in Michigan City, it has a gamma distribution with parameter $(s, \lambda) = (3, 2.5)$.
$f_{X_1+X_2}(4) =$ (circle *none, one or more*)
   i. $\frac{\lambda e^{-\lambda a}(\lambda a)^{s+t-1}}{\Gamma(s+t)} = \frac{2.5e^{-2.5}(2.5(4))^{2+3-1}}{\Gamma(2+3)}$
   ii. $\frac{2.5e^{-(2.5)(4)}(2.5(4))^{2+3-1}}{(5-1)!}$
   iii. 0.0473

(b) For $t = 2$, $s = 3$, $P\{X_1 + X_2 \leq 4\} =$ (circle one) **0.189** / **0.358** / **0.543**.
(Hint: fnInt($\frac{2.5e^{-(2.5)(4)}(2.5(4))^{2+3-1}}{(5-1)!}$, X, 0, 4))

5. *Continuous: normal.* After some effort, it can be shown that the distribution of the sum of independent random variables, $\sum_{i=1}^{n} X_i$, where each X_i has a normal distribution, and where each has the parameter (μ_i, σ_i^2), is also normal with parameter $(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2)$. Consequently, the distribution of $X_1 + X_2$, where X_1 is normal with parameter $(2, 3)$ and X_2 is normal with parameter $(1, 1)$, is normal with parameter (circle one) $\mathbf{(2,3)}$ / $\mathbf{(3,4)}$ / $\mathbf{(4,5)}$.

Review Chapter

Properties of Expectation

7.1 Introduction

If

$$P\{a \le X \le b\} = 1$$

then

$$a \le E(X) \le b$$

Exercise 7.1 (Introduction)

1. Let X be the waiting time for a bus. The chance the waiting time is between 3 and 7 minutes is 100%, $P\{3 \le X \le 7\} = 1$. This means the waiting time is *expected* to between (circle one) $\mathbf{(3, 7)}$ / $\mathbf{(1, 6)}$ / $\mathbf{(4, 8)}$ minutes.

2. Let X be the weight of a new–born child. The chance the weight is between 4 and 10 pounds is 100%, $P\{3 \le X \le 10\} = 1$. This means the weight is expected to between (circle one) $\mathbf{(3, 7)}$ / $\mathbf{(4, 10)}$ / $\mathbf{(4, 8)}$ minutes.

7.2 Expectation of Sums of Random Variables

We look at the expectation of the sums of random variables.

- In general,

$$E[g(x,y)] = \begin{cases} \sum_x \sum_y g(x,y)p(x,y) & \text{if discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)\,dxdy & \text{if continuous} \end{cases}$$

- In particular, if $g(x, y)$ is the sum of random variables,

$$E(X+Y) = E(X) + E(Y) \tag{7.1}$$
$$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] \tag{7.2}$$

Exercise 7.2 (Expectation of Sums of Random Variables)

1. *Expectation, Discrete: Montana Fishing Trip.* The joint density, $P\{X, Y\}$, of the number of minutes waiting to catch the *first* fish, X, *and* the number of minutes waiting to catch the *second* fish, Y, is given below. In addition to the joint distribution probabilities, the average number of fish caught on two trips to the lake, $g(x, y) = \frac{x+y}{2}$, are also given in the *one* table below.

	$P\{X=i, Y=j\}$ $g(x,y) = \frac{x+y}{2}$	j 1	2	3
	1	0.16 1	0.16 $\frac{3}{2}$	0.08 2
i	2	0.16 $\frac{3}{2}$	0.16 2	0.08 $\frac{5}{2}$
	3	0.08 2	0.08 $\frac{5}{2}$	0.04 3

(a) **True** / **False** This is a probability density because the probabilities sum to one.

(b) One fish is caught on the first trip and three fish are caught on the second trip with probability
(circle one) **0.04** / **0.08** / **16**.
The *average* waiting time over these two trips is
$g(1, 3) = \frac{1+3}{2} =$ (circle one) **0.3** / **1.5** / **2**.

(c) Two fish is caught on the first trip and three fish are caught on the second trip with probability
(circle one) **0.04** / **0.08** / **16**.
The *average* waiting time over these two trips is
$g(2, 3) = \frac{2+3}{2} =$ (circle one) **0.3** / **1.5** / **2.5**.

(d) Three fish is caught on the first trip and two fish are caught on the second trip with probability
(circle one) **0.04** / **0.08** / **16**.
The average waiting time over these two trips is
$g(3, 2) = \frac{2+3}{2} =$ (circle one) **0.3** / **1.5** / **2.5**.

(e) The *expected* average waiting time over these two trips is

$$
\begin{aligned}
E[g(x,y)] &= \sum_x \sum_y g(x,y)p(x,y) \\
&= (1)(0.16) + \left(\frac{3}{2}\right)(0.16) + (2)(0.08) \\
&\quad + \left(\frac{3}{2}\right)(0.16) + (2)(0.16) + \left(\frac{3}{2}\right)(0.08) \\
&\quad + (2)(0.08) + \left(\frac{5}{2}\right)(0.08) + (3)(0.04) =
\end{aligned}
$$

(circle one) **1** / **1.5** / **1.8**.

2. *More Expectation, Discrete.*

(a) $g(x,y) = 3xy$. Consider the following joint density. Notice that $g(x,y) = 3xy$.

	$P\{X = x, Y = y\}$ $g(x,y) = 3xy$	1	y 2	3
	1	0.16 3	0.16 6	0.08 9
x	2	0.16 6	0.16 12	0.08 18
	3	0.08 9	0.08 18	0.04 27

The expected value of $g(x,y) = 3xy$ is

$$
\begin{aligned}
E[3XY] &= \sum_x \sum_y 3xy\, p(x,y) \\
&= (3)(0.16) + (6)(0.16) + (9)(0.08) \\
&\quad + (6)(0.16) + (12)(0.16) + (18)(0.08) \\
&\quad + (9)(0.08) + (18)(0.08) + (27)(0.04) =
\end{aligned}
$$

(circle one) **8.4** / **9.72** / **11**.
(Hint: Type $g(x,y)$ into L_1, $P\{X,Y\}$ into L_2, define $L_3 = L_1 \times L_2$, the sum L_3 by STAT CALC 1–Var Stats.)

(b) $g(x,y) = 3xy$. If $g(x,y) = x^2y$, $x = 1,2,3$ and $y = 1,2,3$, then

$$
\begin{aligned}
E[X^2Y] &= \sum_x \sum_y x^2y\, p(x,y) \\
&= (1^2)(1)(0.16) + (1^2)(2)(0.16) + (1^2)(3)(0.08) \\
&\quad + (2^2)(1)(0.16) + (2^2)(2)(0.16) + (2^2)(3)(0.08) \\
&\quad + (3^2)(1)(0.08) + (3^2)(2)(0.08) + (3^2)(3)(0.04) =
\end{aligned}
$$

(circle one) **6.84** / **8.44** / **11.02**.

(c) If $g(x, y) = x/y$, $x = 1, 2, 3$ and $y = 1, 2, 3$, then

$$\begin{aligned} E[X/Y] &= \sum_x \sum_y \frac{x}{y}\, p(x, y) \\ &= (1/1)(0.16) + (1/2)(0.16) + (1/3)(0.08) \\ &\quad + (2/1)(0.16) + (2/2)(0.16) + (2/3)(0.08) \\ &\quad + (3/1)(0.08) + (3/2)(0.08) + (3/3)(0.04) = \end{aligned}$$

(circle one) **0.8** / **1.2** / **2.5**.

(d) If $g(x, y) = x + y$, $x = 1, 2, 3$ and $y = 1, 2, 3$, then

$$\begin{aligned} E[X+Y] &= \sum_x \sum_y (x + y)\, p(x, y) \\ &= (1+1)(0.16) + (1+2)(0.16) + (1+3)(0.08) \\ &\quad + (2+1)(0.16) + (2+2)(0.16) + (2+3)(0.08) \\ &\quad + (3+1)(0.08) + (3+2)(0.08) + (3+3)(0.04) = \end{aligned}$$

(circle one) **0.8** / **1.2** / **3.6**.

3. *More Expectation, Discrete.* Consider the following joint density.

$P\{X = i, Y = j\}$		j = 1	2	3	row sum $P\{X = i\}$
i	1	0.01	0.02	0.08	0.11
	2	0.01	0.02	0.08	0.11
	3	0.07	0.08	0.63	0.78
$P\{Y = j\}$ column sum		0.09	0.12	0.79	

(a) The expected value of $g(x, y) = 3xy$ is

$$\begin{aligned} E[3XY] &= \sum_x \sum_y 3xy\, p(x, y) \\ &= 3(1)(1)(0.01) + 3(1)(2)(0.02) + 3(1)(3)(0.08) \\ &\quad + 3(2)(1)(0.01) + 3(2)(2)(0.02) + 3(2)(3)(0.08) \\ &\quad + 3(3)(1)(0.07) + 3(3)(2)(0.08) + 3(3)(3)(0.63) = \end{aligned}$$

(circle one) **18.43** / **21.69** / **11.22**.

(b) The expected value of $g(x, y) = x + y$ is

$$\begin{aligned} E[X+Y] &= \sum_x \sum_y (x+y)\, p(x,y) \\ &= (1+1)(0.01) + (1+2)(0.02) + (1+3)(0.08) \\ &\quad + (2+1)(0.01) + (2+2)(0.02) + (2+3)(0.08) \\ &\quad + (3+1)(0.07) + (3+2)(0.08) + (3+3)(0.63) \; = \end{aligned}$$

(circle one) **5.37** / **8.74** / **11.2**.

(c) The expected value of $g(x, y) = x$ is

$$\begin{aligned} E[X] &= \sum_x \sum_y x\, p(x,y) \\ &= (1)(0.01) + (1)(0.02) + (1)(0.08) \\ &\quad + (2)(0.01) + (2)(0.02) + (2)(0.08) \\ &\quad + (3)(0.07) + (3)(0.08) + (3)(0.63) \\ &= (1)(0.11) + (2)(0.11) + (3)(0.78) \; = \\ &= \sum_x x p_X(x) \; = \end{aligned}$$

(circle one) **2.67** / **8.74** / **11.2**.

(d) The expected value of $g(x, y) = y$ is

$$\begin{aligned} E[X] &= \sum_y y p_Y(y) \\ &= (1)(0.09) + (2)(0.12) + (3)(0.79) \; = \end{aligned}$$

(circle one) **2.67** / **2.7** / **11.2**.

(e) **True** / **False** $E(X+Y) = 5.37 = 2.67 + 2.7 = E(X) + E(Y)$

4. *Expectation, Discrete.* Let

$$p(x, y, z) = \frac{1}{24}, \; x = 1, 2; \; y = 1, 2, 3; \; z = 1, 2, 3, 4$$

and

$$g(x, y, z) = x + y + z$$

(a) $p_X(1) = \sum_{y=1}^{3} \sum_{z=1}^{4} p(1, y, z) =$ (circle one) $\mathbf{\frac{12}{24}}$ / $\mathbf{\frac{13}{24}}$ / $\mathbf{\frac{14}{24}}$.

(b) $p_X(2) = \sum_{y=1}^{3} \sum_{z=1}^{4} p(2, y, z) =$ (circle one) $\mathbf{\frac{12}{24}}$ / $\mathbf{\frac{13}{24}}$ / $\mathbf{\frac{14}{24}}$.

(c) The expected value of $g(x,y,z) = x$ is

$$\begin{aligned} E[X] &= \sum_x x\, p_X(x) \\ &= (1)\left(\frac{12}{24}\right) + (2)\left(\frac{12}{24}\right) = \end{aligned}$$

(circle one) **1** / **1.5** / **2.5**.

(d) The expected value of $g(x,y,z) = y$ is

$$\begin{aligned} E[Y] &= \sum_y y\, p_Y(y) \\ &= (1)\left(\frac{8}{24}\right) + (2)\left(\frac{8}{24}\right) + (3)\left(\frac{8}{24}\right) = \end{aligned}$$

(circle one) **1** / **1.5** / **2**.

(e) The expected value of $g(x,y,z) = z$ is

$$\begin{aligned} E[Z] &= \sum_z z\, p_Z(z) \\ &= (1)\left(\frac{6}{24}\right) + (2)\left(\frac{6}{24}\right) + (3)\left(\frac{6}{24}\right) + (4)\left(\frac{6}{24}\right) = \end{aligned}$$

(circle one) **1** / **1.5** / $\mathbf{\frac{60}{24}}$.

(f) $E(X+Y+Z) = E(X) + E(Y) + E(Z) =$ (circle one) **6** / **8.74** / **11.2**.

5. *Expectation, Continuous.* Consider the following distribution,

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0 < x < 1, 0 < y < 2 \\ 0 & elsewhere \end{cases}$$

(a) $f_X(x) =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3}\right)\, dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right)\, dy$

ii. $\left(yx^2 + \frac{xy^2}{6}\right)_0^2$

iii. $2x^2 + \frac{2x}{3}$

(b) $E(X) =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} x\left(2x^2 + \frac{2x}{3}\right)\, dx = \int_0^1 \left(2x^3 + \frac{2x^2}{3}\right)\, dx$

ii. $\left(\frac{2}{4}x^4 + \frac{2x^3}{9}\right)_0^1$

iii. $\frac{2}{4} + \frac{2}{9} = \frac{13}{18}$

(c) $f_Y(y) =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3}\right)\, dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right)\, dx$

ii. $\left(\frac{1}{3}x^3 + \frac{x^2y}{6}\right)_0^1$

iii. $\frac{1}{3} + \frac{y}{6}$

(d) $E(Y) =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} y\left(\frac{1}{3} + \frac{y}{6}\right)\, dy = \int_0^2 \left(\frac{y}{3} + \frac{y^2}{6}\right)\, dy$

ii. $\left(\frac{1}{6}y^2 + \frac{y^3}{18}\right)_0^2$

iii. $\frac{1}{6}(2)^2 + \frac{2^3}{18} = \frac{10}{9}$

(e) $E(X+Y) = E(X) + E(Y) = \frac{13}{18} + \frac{10}{9} =$ (circle one) $\mathbf{\frac{11}{6}}$ / **3** / **4**.

(f) $E(X+Y) =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)\, dx\, dy = \int_0^2\int_0^1(x+y)\left(x^2 + \frac{xy}{3}\right)\, dx\, dy$

ii. $\int_0^2\int_0^1 \left(x^3 + \frac{4x^2y}{3} + \frac{xy^2}{3}\right)\, dx\, dy$

iii. $\int_0^2 \left(\frac{x^4}{4} + \frac{4x^3y}{9} + \frac{x^2y^2}{6}\right)_0^1 dy$

iv. $\int_0^2 \left(\frac{1}{4} + \frac{4y}{9} + \frac{y^2}{6}\right)\, dy$

v. $\left(\frac{y}{4} + \frac{4y^2}{18} + \frac{y^3}{18}\right)_0^2 \;=\; \frac{11}{6}$

6. *Expectation of Binomial.* Let X be a binomial random variable with parameters n and p and let

$$X = X_1 + X_2 + \cdots + X_n$$

where

$$X_i = \begin{cases} 1 & \text{if } i\text{th trial is a success} \\ 0 & \text{if } i\text{th trial is a failure} \end{cases}$$

(a) Each X_i is a Bernoulli where
$E(X_i) = 1(p) + 0(1-p) =$ (circle one) $\boldsymbol{p}$ / $\boldsymbol{1-p}$ / **1**.

(b) And so $E(X) = E(X_1) + \cdots + E(X_n) =$ (circle one) $\boldsymbol{np}$ / $\boldsymbol{n(1-p)}$ / $\boldsymbol{n}$.

(c) If $n = 8$ and $p = 0.25$, then $E(X) =$ (circle one) **3** / **4** / **5**.

7. *Expected Number of Matches.* Ten people throw ten tickets with their names on each ticket into a jar, then draw one ticket out of the jar at random (and put it back in the jar). Let X be the number of people who select their own ticket out of the jar. Let

$$X = X_1 + X_2 + \cdots + X_{10}$$

where

$$X_i = \begin{cases} 1 & \text{if } i\text{th person selects own ticket} \\ 0 & \text{if } i\text{th person does not select their own ticket} \end{cases}$$

(a) Since each person will choose any ticket with equal chance, $E(X_i) = P\{X_i = 1\} =$ (circle one) **0.1** / **0.2** / **0.3**.

(b) And so $E(X) = E(X_1) + \cdots + E(X_{10}) = 10(0.1) =$ (circle one) **1** / **5** / **10**. In other words, we'd expect one of the ten individuals to choose their own ticket.

(c) *If*, instead of ten individuals, n individuals played this game, then we would expect $E(X) = E(X_1) + \cdots + E(X_n) = n\frac{1}{n}$ (circle one) **1** / **5** / **10**.

7.3 Covariance, Variance of Sums and Correlations

In this section, we look at covariance, a measure of how two random variables are related.

- If X, Y independent, then $E[g(x)h(Y)] = E(g(X))E(h(Y))$
- Covariance is defined by
  $\text{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$
  and has the following properties.
  - $\text{Cov}(X,Y) = \text{Cov}(Y,X)$
  - $\text{Cov}(X,X) = \text{Var}(X)$
  - $\text{Cov}(aX,Y) = a\text{Cov}(X,Y)$
  - $\text{Cov}(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, Y_j)$
- In general,
  $\text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, Y_j)$,
- The correlation is given by
  $\rho(x,y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$

Exercise 7.3 (Covariance, Variance of Sums and Correlations)

1. *Covariance, Discrete.* Consider the following joint density.

$P\{X=i,Y=j\}$		j 1	2	3	row sum $P\{X=i\}$
	1	0.01	0.02	0.08	0.11
i	2	0.01	0.02	0.08	0.11
	3	0.07	0.08	0.63	0.78
$P\{Y=j\}$ column sum		0.09	0.12	0.79	

(a) The expected value of $g(x,y) = xy$ is

$$\begin{aligned} E[XY] &= \sum_x \sum_y xy\, p(x,y) \\ &= (1)(1)(0.01) + (1)(2)(0.02) + (1)(3)(0.08) \\ &\quad + (2)(1)(0.01) + (2)(2)(0.02) + (2)(3)(0.08) \\ &\quad + (3)(1)(0.07) + (3)(2)(0.08) + (3)(3)(0.63) \ = \end{aligned}$$

(circle one) **7.23** / **13.74** / **11.22**.

(b) The expected value of $g(x,y) = x$ is

$$\begin{aligned} E[X] &= \sum_x x\, p_X(x) \ = \\ &= (1)(0.11) + (2)(0.11) + (3)(0.78) \ = \end{aligned}$$

(circle one) **2.67** / **8.74** / **11.2**.

(c) The expected value of $g(x,y) = y$ is

$$\begin{aligned} E[Y] &= \sum_y y\, p_Y(y) \\ &= (1)(0.09) + (2)(0.12) + (3)(0.79) \ = \end{aligned}$$

(circle one) **2.67** / **2.7** / **11.2**.

(d) $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 7.23 - (2.67)(2.7) =$ (circle one) **0.021** / **0.335** / **0.545**.

(e) $\text{Cov}(Y,X) = E(YX) - E(Y)E(X) = 7.23 - (2.7)(2.67) =$ (circle one) **0.021** / **0.335** / **0.545**.

(f) **True** / **False**. $\text{Cov}(Y,X) = \text{Cov}(X,Y)$

(g) The expected value of $g(x,y) = x^2$ is

$$\begin{aligned} E[X^2] &= \sum_x x^2\, p_X(x) \ = \\ &= (1^2)(0.11) + (2^2)(0.11) + (3^2)(0.78) \ = \end{aligned}$$

(circle one) **2.67** / **7.57** / **11.2**.

(h) $\text{Cov}(X, X) = \text{Var}(X) = E(X^2) - E(X)E(X) = 7.57 - (2.67)(2.67) =$ (circle one) **0.4411** / **7.57** / **11.2**.

(i) The expected value of $g(x, y) = y^2$ is

$$\begin{aligned} E[Y^2] &= \sum_y y^2 \, p_Y(y) = \\ &= (1^2)(0.09) + (2^2)(0.12) + (3^2)(0.79) = \end{aligned}$$

(circle one) **2.67** / **7.57** / **7.68**.

(j) $\text{Var}(Y) = E(Y^2) - [E(Y)]^2 =$ (circle one) **0.39** / **0.5511** / **11.2**.

(k) $\text{Cov}(3X, Y) = 3 \text{ Cov}(XY) = 3(0.021) =$ (circle one) **0.021** / **0.063** / **0.545**.

(l) $\text{Cov}(X, 4Y) = 4 \text{ Cov}(XY) = 4(0.021) =$ (circle one) **0.021** / **0.063** / **0.084**.

(m) $\text{Cov}(3X, 4Y) = (3)(4) \text{ Cov}(XY) = 12(0.021) =$ (circle one) **0.021** / **0.063** / **0.252**.

(n) **True** / **False**. If X, Y independent, then $E[g(x)h(Y)] = E(g(X))E(h(Y))$. *If* $E(XY) = E(X)E(Y)$, then $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$. In other words, if X and Y are independent, then the covariance of X and Y is zero. (The converse is not necessarily true.)

(o) *Correlation.* $\rho(x, y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{0.021}{\sqrt{(0.4411)(0.39)}} =$ (circle one) **0.021** / **0.053** / **0.084**.
Correlation measure how linear the data is; $\rho = 1$ (positive) if $Y = aX + b$ where $a > 0$ and $\rho = -1$ (negative) if $Y = aX + b$ where $a < 0$ and $\rho = 0$ (uncorrelated) if $Y = aX + b$ where $a = 0$.

2. *More Covariance, Discrete.* Let

$$p(x, y, z) = \frac{1}{24}, \; x = 1, 2; \; y = 1, 2, 3; \; z = 1, 2, 3, 4$$

and

$$g(x, y, z) = x + y + z$$

(a) The expected value of $g(x, y, z) = x$ is

$$\begin{aligned} E[X] &= \sum_x x \, p_X(x) \\ &= (1)\left(\frac{12}{24}\right) + (2)\left(\frac{12}{24}\right) = \end{aligned}$$

(circle one) **1** / **1.5** / **2.5**.

(b) The expected value of $g(x, y, z) = y$ is

$$\begin{aligned} E[Y] &= \sum_y y\, p_Y(y) \\ &= (1)\left(\frac{8}{24}\right) + (2)\left(\frac{8}{24}\right) + (3)\left(\frac{8}{24}\right) = \end{aligned}$$

(circle one) **1** / **1.5** / **2**.

(c) The expected value of $g(x, y, z) = z$ is

$$\begin{aligned} E[Z] &= \sum_z z\, p_Z(z) \\ &= (1)\left(\frac{6}{24}\right) + (2)\left(\frac{6}{24}\right) + (3)\left(\frac{6}{24}\right) + (4)\left(\frac{6}{24}\right) = \end{aligned}$$

(circle one) **1** / **1.5** / $\mathbf{\frac{60}{24}}$.

(d) The expected value of $g(x, y) = xy$ is

$$\begin{aligned} E[XY] &= \sum_x \sum_y xy\, p_{X,Y}(x, y) \\ &= (1)(1)\left(\frac{4}{24}\right) + (1)(2)\left(\frac{4}{24}\right) + (1)(3)\left(\frac{4}{24}\right) \\ &\quad + (2)(1)\left(\frac{4}{24}\right) + (2)(2)\left(\frac{4}{24}\right) + (2)(3)\left(\frac{4}{24}\right) = \end{aligned}$$

(circle one) $\mathbf{\frac{71}{24}}$ / $\mathbf{\frac{72}{24}}$ / $\mathbf{\frac{73}{24}}$.

(e) The expected value of $g(x, z) = xz$ is

$$\begin{aligned} E[XZ] &= \sum_x \sum_z xz\, p_{X,Z}(x, z) \\ &= (1)(1)\left(\frac{3}{24}\right) + (1)(2)\left(\frac{3}{24}\right) + (1)(3)\left(\frac{3}{24}\right) + (1)(4)\left(\frac{3}{24}\right) \\ &\quad + (2)(1)\left(\frac{3}{24}\right) + (2)(2)\left(\frac{3}{24}\right) + (2)(3)\left(\frac{3}{24}\right) + (2)(4)\left(\frac{3}{24}\right) = \end{aligned}$$

(circle one) $\mathbf{\frac{71}{24}}$ / $\mathbf{\frac{72}{24}}$ / $\mathbf{\frac{90}{24}}$.

(f) $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{72}{24} - (1.5)(2) =$
(circle one) **0** / **0.335** / **0.545**.

(g) $\text{Cov}(X, Z) = E(XZ) - E(X)E(Z) = \frac{90}{24} - (1.5)\left(\frac{60}{24}\right) =$
(circle one) **0** / **0.335** / **0.545**.

(h) Since $\text{Cov}(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, Y_j)$,
$\text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z) = 0 + 0 =$
(circle one) **0** / **0.335** / **0.545**.

(i) Since $\text{Cov}(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, Y_j)$,
$\text{Cov}(X, Y+Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z) = 0 + 0 =$
(circle one) **0** / **0.335** / **0.545**.

(j) The expected value of $g(x, y, z) = x^2$ is

$$\begin{aligned} E[X^2] &= \sum_x x^2 \, p_X(x) \\ &= (1)^2 \left(\frac{12}{24}\right) + (2)^2 \left(\frac{12}{24}\right) = \end{aligned}$$

(circle one) $\mathbf{\frac{60}{24}}$ / **1.5** / **2.5**.

(k) $\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{60}{24} - (1.5)^2 =$ (circle one) **0** / **0.25** / **0.545**.

(l) The expected value of $g(x, y, z) = y^2$ is

$$\begin{aligned} E[Y^2] &= \sum_y y^2 \, p_Y(y) \\ &= (1)^2 \left(\frac{8}{24}\right) + (2)^2 \left(\frac{8}{24}\right) + (3)^2 \left(\frac{8}{24}\right) = \end{aligned}$$

(circle one) **1** / **1.5** / $\mathbf{\frac{112}{24}}$.

(m) $\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{112}{24} - (2)^2 =$ (circle one) **0** / **0.5** / $\mathbf{\frac{2}{3}}$.

(n) Since $\text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i) + 2\underline{\sum\sum}_{i<j} \text{Cov}(X_i, Y_j)$,
$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X, Y) = 0.25 + \frac{2}{3} + 2(0) =$
(circle one) $\mathbf{\frac{9}{12}}$ / $\mathbf{\frac{10}{12}}$ / $\mathbf{\frac{11}{12}}$.

3. *Variance of Rolling Dice.* Calculate the expectation and variance of the sum of 15 rolls of a fair die.

(a) For the ith roll, $X_i = 1, 2, 3, 4, 5, 6$,
$E[X_i] = \sum_{j=1}^6 xp(x) = 1(1/6) + 2(1/6) + \cdots + 6(1/6) =$
(circle one) $\mathbf{\frac{3}{2}}$ / $\mathbf{\frac{5}{2}}$ / $\mathbf{\frac{7}{2}}$.

(b) and so

$$\begin{aligned} E\left(\sum_{i=1}^{15} X_i\right) &= \sum_{i=1}^{10} E[X_i] \\ &= 15E[X_i] \\ &= 15\frac{7}{2} = \end{aligned}$$

(circle one) $\mathbf{\frac{75}{2}}$ / $\mathbf{\frac{90}{2}}$ / $\mathbf{\frac{105}{2}}$.

(c) Since $E[X_i^2] = \sum_{j=1}^{6} x^2 p(x) = 1^2(1/6) + 2^2(1/6) + \cdots + 6^2(1/6) = \frac{91}{6}$
(circle one) $\mathbf{\frac{88}{6}}$ / $\mathbf{\frac{91}{6}}$ / $\mathbf{\frac{95}{6}}$.
and so $\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = 91/6 - (7/2)^2 =$,
(circle one) $\mathbf{\frac{30}{12}}$ / $\mathbf{\frac{35}{12}}$ / $\mathbf{\frac{40}{12}}$.

(d) Since X_i are independent,

$$\begin{aligned} V\left(\sum_{i=1}^{10} X_i\right) &= \sum_{i=1}^{10} V[X_i] \\ &= 15V[X_i] \\ &= 15\frac{35}{12} = \end{aligned}$$

(circle one) $\mathbf{\frac{165}{4}}$ / $\mathbf{\frac{170}{4}}$ / $\mathbf{\frac{175}{4}}$.

4. *Covariance, Continuous.* Consider the following distribution,

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0 < x < 1, 0 < y < 2 \\ 0 & elsewhere \end{cases}$$

(a) $f_X(x) =$ (circle none, one or more)
   i. $\int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3}\right)\, dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right)\, dy$
   ii. $\left(yx^2 + \frac{xy^2}{6}\right)_0^2$
   iii. $2x^2 + \frac{2x}{3}$

(b) $E(X) =$ (circle none, one or more)
   i. $\int_{-\infty}^{\infty} x\left(2x^2 + \frac{2x}{3}\right)\, dx = \int_0^1 \left(2x^3 + \frac{2x^2}{3}\right)\, dx$
   ii. $\left(\frac{2}{4}x^4 + \frac{2x^3}{9}\right)_0^1$
   iii. $\frac{2}{4} + \frac{2}{9} = \frac{13}{18}$

(c) $f_Y(y) =$ (circle none, one or more)
   i. $\int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3}\right)\, dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right)\, dx$
   ii. $\left(\frac{1}{3}x^3 + \frac{x^2y}{6}\right)_0^1$
   iii. $\frac{1}{3} + \frac{y}{6}$

(d) $E(Y) =$ (circle none, one or more)
   i. $\int_{-\infty}^{\infty} y\left(\frac{1}{3} + \frac{y}{6}\right)\, dy = \int_0^2 \left(\frac{y}{3} + \frac{y^2}{6}\right)\, dy$
   ii. $\left(\frac{1}{6}y^2 + \frac{y^3}{18}\right)_0^2$
   iii. $\frac{1}{6}(2)^2 + \frac{2^3}{18} = \frac{10}{9}$

(e) $E(XY) =$

i. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(xy)f(x,y)\,dx\,dy = \int_0^2\int_0^1 (xy)\left(x^2+\frac{xy}{3}\right)\,dx\,dy$

ii. $\int_0^2\int_0^1 \left(x^3y+\frac{x^2y^2}{3}\right)\,dx\,dy$

iii. $\int_0^2 \left(\frac{x^4y}{4}+\frac{x^3y^2}{9}\right)_0^1 dy$

iv. $\int_0^2 \left(\frac{y}{4}+\frac{y^2}{9}\right)\,dy$

v. $\left(\frac{y^2}{8}+\frac{y^3}{27}\right)_0^2 =$

(circle one) $\mathbf{\frac{43}{54}}$ / $\mathbf{\frac{44}{54}}$ / $\mathbf{\frac{45}{54}}$.

(f) $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{43}{54} - \left(\frac{13}{18}\right)\left(\frac{10}{9}\right) =$
(circle one) $\mathbf{-0.0062}$ / $\mathbf{0.0062}$ / $\mathbf{\frac{3}{18}}$.

(g) $E(X^2) =$

i. $\int_{-\infty}^{\infty} x^2\left(2x^2+\frac{2x}{3}\right)\,dx = \int_0^1 \left(2x^4+\frac{2x^3}{3}\right)\,dx$

ii. $\left(\frac{2}{5}x^5+\frac{2x^4}{12}\right)_0^1$

iii. $\frac{2}{5}+\frac{2}{12} =$

(circle one) $\mathbf{\frac{16}{30}}$ / $\mathbf{\frac{17}{30}}$ / $\mathbf{\frac{18}{30}}$.

(h) $\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{17}{30} - \left(\frac{13}{18}\right)^2 =$ (circle one) $\mathbf{\frac{73}{1620}}$ / $\mathbf{\frac{17}{30}}$ / $\mathbf{\frac{18}{30}}$.

(i) $E(Y^2) =$

i. $\int_{-\infty}^{\infty} y^2\left(\frac{1}{3}+\frac{y}{6}\right)\,dy = \int_0^2 \left(\frac{y^2}{3}+\frac{y^3}{6}\right)\,dy$

ii. $\left(\frac{1}{9}y^3+\frac{y^4}{24}\right)_0^2$

iii. $\frac{1}{9}(2)^3+\frac{2^4}{24} =$

(circle one) $\mathbf{\frac{13}{9}}$ / $\mathbf{\frac{14}{9}}$ / $\mathbf{\frac{15}{9}}$.

(j) $\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{14}{9} - \left(\frac{10}{9}\right)^2 =$ (circle one) $\mathbf{\frac{26}{81}}$ / $\mathbf{\frac{17}{30}}$ / $\mathbf{\frac{18}{30}}$.

(k) $\text{Cov}(5X,Y) = 5\ \text{Cov}(XY) = 5(-0.0062) =$
(circle one) $\mathbf{-0.021}$ / $\mathbf{-0.031}$ / $\mathbf{-0.123}$.

(l) $\text{Cov}(X,4Y) = 4\ \text{Cov}(XY) = 4(-0.0062) =$
(circle one) $\mathbf{-0.021}$ / $\mathbf{-0.063}$ / $\mathbf{-0.123}$.

(m) *Correlation.* $\rho(x,y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{V}ar(X)\text{V}ar(Y)}} = \frac{-0.0062}{\sqrt{(73/1620)(26/81)}} =$
(circle one) $\mathbf{-0.024}$ / $\mathbf{-0.052}$ / $\mathbf{-0.084}$.

5. *Covariance of Binomial.* Let X be a binomial random variable with parameters n and p and let

$$X = X_1 + X_2 + \cdots + X_n$$

where

$$X_i = \begin{cases} 1 & \text{if } i\text{th trial is a success} \\ 0 & \text{if } i\text{th trial is a failure} \end{cases}$$

(a) Each X_i is a Bernoulli where
$E(X_i) = 1(p) + 0(1-p) =$ (circle one) $\boldsymbol{p}$ / $\boldsymbol{1-p}$ / $\mathbf{1}$.

(b) $E(X_i^2) = 1^2(p) + 0^2(1-p) =$ (circle one) $\boldsymbol{p}$ / $\boldsymbol{1-p}$ / $\mathbf{1}$.
In other words, $E(X_i^2) = E(X_i)$; in fact, $E(X_i^n) = E(X_i)$.

(c) $\text{Var}(X_i) = E(X_i^2) - [E(X_i)]^2 = p - (p)^2 =$ (circle one) $\boldsymbol{p}$ / $\boldsymbol{p(1-p)}$ / $\mathbf{1}$.

(d) For $i \neq j$, $E(X_iX_j) = (1)(1)(p)(p) + (1)(0)(p)(1-p) + (0)(1)(1-p)(p) + 0^2(1-p)^2 =$ (circle one) $\boldsymbol{p}$ / $\boldsymbol{1-p}$ / $\boldsymbol{p^2}$.

(e) $\text{Cov}(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) = p^2 - (p)(p) =$
(circle one) $\mathbf{0}$ / $\mathbf{0.0062}$ / $\mathbf{\frac{3}{18}}$.

(f) Since $\text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, Y_j)$, and $\text{Cov}(X_i, X_j) = 0$,
$\text{Var}(X_1 + \cdots + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n)$
$= p(1-p) + \cdots + p(1-p) =$ (circle one) $\boldsymbol{p(1-p)}$ / $\boldsymbol{np(1-p)}$ / $\mathbf{1}$.

7.4 Conditional Expectation

We learn about conditional expectation and variance; in particular, we look at the following items.

- $E[X|Y=y] = \sum_x xP\{X=x|Y=y\} = \sum xp_{X|Y}(x|y)$, discrete
  $E[X|Y=y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)\,dx$, $f_X(x) > 0$, continuous
- $E[g(X)|Y=y] = \sum g(x)p_{X|Y}(x|y)$, discrete
  $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)\,dx$, $f_X(x) > 0$, continuous
- general sum: $E[\sum_{i=1}^n X_i|Y=y] = \sum_{i=1}^n E(X_i|Y=y)$
- computing expectation by conditioning,
  $E[X] = E[E(X|Y)] = \sum_y E(X=x|Y=y)P(Y=y)$, discrete
  $E[X] = \int_{-\infty}^{\infty} E(X=x|Y=y)f_Y(y)\,dy$, continuous
- computing probability by conditioning,
  if $E(X) = P(E)$, where $X = 1$ if E occurs, 0 otherwise
  then $E[X] = \sum_y P(E|Y=y)P(Y=y)$, discrete (generalization of total probability)
  $E[X] = \int_{-\infty}^{\infty} P(E|Y=y)f_Y(y)$, continuous

- conditional variance,
  $\mathrm{Var}(X|Y=y) = E[(X - E(X|Y=y))^2|Y=y] = E(X^2|Y=y) - (E(X|Y=y))^2$,
  $\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E(X|Y))$

Exercise 7.4 (Conditional Expectation)

1. *Conditional Expectation, Discrete.* Consider the following joint density.

$P\{X=i, Y=j\}$		j 1	2	3	row sum $P\{X=i\}$
	1	0.01	0.02	0.08	0.11
i	2	0.01	0.02	0.08	0.11
	3	0.07	0.08	0.63	0.78
$P\{Y=j\}$ column sum		0.09	0.12	0.79	

   (a) *Compute $E[Y|X=2]$.*
   Since $P\{Y=1|X=2\} = P\{Y=1, X=2\}/P\{X=2\} = \frac{0.01}{0.11} =$
   (circle one) $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{2}{11}}$ / $\mathbf{\frac{7}{11}}$ / $\mathbf{\frac{8}{11}}$
   and $P\{Y=2|X=2\} = P\{Y=2, X=2\}/P\{X=2\} = \frac{0.02}{0.11} =$
   (circle one) $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{2}{11}}$ / $\mathbf{\frac{7}{11}}$ / $\mathbf{\frac{8}{11}}$
   and $P\{Y=3|X=2\} = P\{Y=3, X=2\}/P\{X=2\} = \frac{0.08}{0.11} =$
   (circle one) $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{2}{11}}$ / $\mathbf{\frac{7}{11}}$ / $\mathbf{\frac{8}{11}}$
   then $E[Y|X=2] = \sum_y yP\{Y=y|X=2\} = (1)\left(\frac{1}{11}\right) + (2)\left(\frac{2}{11}\right) + (3)\left(\frac{8}{11}\right) =$
   (circle one) $\mathbf{\frac{1}{11}}$ / $\mathbf{\frac{2}{11}}$ / $\mathbf{\frac{29}{11}}$ / $\mathbf{\frac{30}{11}}$

   (b) *Compute $E[X|Y=1]$.*
   Since $P\{X=1|Y=1\} = P\{X=1, Y=1\}/P\{Y=1\} = \frac{0.01}{0.09} =$
   (circle one) $\mathbf{\frac{1}{9}}$ / $\mathbf{\frac{2}{9}}$ / $\mathbf{\frac{7}{9}}$ / $\mathbf{\frac{8}{9}}$
   and $P\{X=2|Y=1\} = P\{X=2, Y=1\}/P\{Y=1\} = \frac{0.01}{0.09} =$
   (circle one) $\mathbf{\frac{1}{9}}$ / $\mathbf{\frac{2}{9}}$ / $\mathbf{\frac{7}{9}}$ / $\mathbf{\frac{8}{9}}$
   and $P\{X=3|Y=1\} = P\{X=3, Y=1\}/P\{Y=1\} = \frac{0.07}{0.09} =$
   (circle one) $\mathbf{\frac{1}{9}}$ / $\mathbf{\frac{2}{9}}$ / $\mathbf{\frac{7}{9}}$ / $\mathbf{\frac{8}{9}}$
   then $E[X|Y=1] = \sum_x xP\{X=x|Y=1\} = (1)\left(\frac{1}{9}\right) + (2)\left(\frac{1}{9}\right) + (3)\left(\frac{7}{9}\right) =$
   (circle one) $\mathbf{\frac{1}{9}}$ / $\mathbf{\frac{2}{9}}$ / $\mathbf{\frac{7}{9}}$ / $\mathbf{\frac{24}{9}}$

   (c) *Compute $E[X|Y=2]$.*
   Since $P\{X=1|Y=2\} = P\{X=1, Y=2\}/P\{Y=2\} = \frac{0.02}{0.12} =$
   (circle one) $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{2}{12}}$ / $\mathbf{\frac{7}{12}}$ / $\mathbf{\frac{8}{12}}$
   and $P\{X=2|Y=2\} = P\{X=2, Y=2\}/P\{Y=2\} = \frac{0.02}{0.12} =$
   (circle one) $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{2}{12}}$ / $\mathbf{\frac{7}{12}}$ / $\mathbf{\frac{8}{12}}$

and $P\{X=3|Y=2\} = P\{X=3, Y=2\}/P\{Y=2\} = \frac{0.08}{0.12} =$
(circle one) $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{2}{12}}$ / $\mathbf{\frac{7}{12}}$ / $\mathbf{\frac{8}{12}}$
then $E[X|Y=2] = \sum_x xP\{X=x|Y=2\} = (1)\left(\frac{2}{12}\right) + (2)\left(\frac{2}{12}\right) + (3)\left(\frac{8}{12}\right) =$
(circle one) $\mathbf{\frac{1}{12}}$ / $\mathbf{\frac{2}{12}}$ / $\mathbf{\frac{27}{12}}$ / $\mathbf{\frac{29}{12}}$

(d) *Compute $E[X|Y=3]$.*
Since $P\{X=1|Y=3\} = P\{X=1, Y=3\}/P\{Y=3\} = \frac{0.08}{0.79} =$
(circle one) $\mathbf{\frac{1}{79}}$ / $\mathbf{\frac{2}{79}}$ / $\mathbf{\frac{7}{79}}$ / $\mathbf{\frac{8}{79}}$
and $P\{X=2|Y=3\} = P\{X=2, Y=3\}/P\{Y=3\} = \frac{0.08}{0.79} =$
(circle one) $\mathbf{\frac{1}{79}}$ / $\mathbf{\frac{2}{79}}$ / $\mathbf{\frac{7}{79}}$ / $\mathbf{\frac{8}{79}}$
and $P\{X=3|Y=3\} = P\{X=3, Y=3\}/P\{Y=3\} = \frac{0.63}{0.79} =$
(circle one) $\mathbf{\frac{1}{79}}$ / $\mathbf{\frac{2}{79}}$ / $\mathbf{\frac{7}{79}}$ / $\mathbf{\frac{63}{79}}$
then $E[X|Y=3] = \sum_x xP\{X=x|Y=3\} = (1)\left(\frac{8}{79}\right) + (2)\left(\frac{8}{79}\right) + (3)\left(\frac{63}{79}\right) =$
(circle one) $\mathbf{\frac{1}{79}}$ / $\mathbf{\frac{2}{79}}$ / $\mathbf{\frac{213}{79}}$ / $\mathbf{\frac{214}{79}}$

(e) *Compare $E[E[X|Y]]$ to $E[X]$.*
Since $E[E(X|Y)] =$ (circle none, one or more)

i. $\sum_y E(X|Y=y)P(Y=y) = E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3)$
ii. $\left(\frac{24}{9}\right)(0.09) + \left(\frac{29}{12}\right)(0.12) + \left(\frac{213}{79}\right)(0.79)$
iii. 2.67

and $E[X] =$ (circle none, one or more)

i. $\sum_x xP(X=x) = (1)P(X=1) + (2)P(X=2) + (3)P(X=3)$
ii. $(1)(0.11) + (2)(0.11) + (3)(0.78)$
iii. 2.67

In other words, $E[X]$ (circle one) **does** / **does not** equal $E[E(X|Y)]$.

(f) *Compute $E(e^X|Y=1)$.*
$E[e^X|Y=1] = \sum_x e^x P\{X=x|Y=1\} = e^1\left(\frac{1}{9}\right) + e^2\left(\frac{1}{9}\right) + e^3\left(\frac{7}{9}\right) =$
(circle one) **15.4** / **15.7** / **16.3** / **16.7**

(g) *Compute $E(X^2|Y=1)$.*
$E[X^2|Y=1] = \sum_x x^2 P\{X=x|Y=1\} = (1)^2\left(\frac{1}{9}\right) + (2)^2\left(\frac{1}{9}\right) + (3)^2\left(\frac{7}{9}\right) =$
(circle one) $\mathbf{\frac{1}{9}}$ / $\mathbf{\frac{2}{9}}$ / $\mathbf{\frac{67}{9}}$ / $\mathbf{\frac{68}{9}}$

(h) *Compute Var$(X|Y=1)$.*
$\text{Var}(X|Y=1) = E(X^2|Y=1) - (E(X|Y=1))^2 = \frac{68}{9} - \left(\frac{24}{9}\right)^2 =$
(circle one) $\mathbf{\frac{1}{9}}$ / $\mathbf{\frac{3}{9}}$ / $\mathbf{\frac{4}{9}}$ / $\mathbf{\frac{11}{9}}$

(i) *Compute Var$(X|Y=2)$.*
Since $E[X^2|Y=2] = \sum_x x^2 P\{X=x|Y=2\} = (1)^2\left(\frac{2}{12}\right) + (2)^2\left(\frac{2}{12}\right) + (3)^2\left(\frac{8}{12}\right) =$

(circle one) $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{41}{6}}$ / $\mathbf{\frac{68}{6}}$
$\text{Var}(X|Y=2) = E(X^2|Y=2) - (E(X|Y=2))^2 = \frac{41}{6} - \left(\frac{29}{12}\right)^2 =$
(circle one) $\mathbf{\frac{1}{144}}$ / $\mathbf{\frac{143}{144}}$ / $\mathbf{\frac{247}{144}}$ / $\mathbf{\frac{368}{144}}$

(j) *Compute Var*$(X|Y=3)$.
Since $E[X^2|Y=3] = \sum_x x^2 P\{X=x|Y=3\} = (1)^2\left(\frac{8}{79}\right) + (2)^2\left(\frac{8}{79}\right) + (3)^2\left(\frac{63}{79}\right) =$
(circle one) $\mathbf{\frac{1}{79}}$ / $\mathbf{\frac{2}{79}}$ / $\mathbf{\frac{67}{79}}$ / $\mathbf{\frac{607}{79}}$
$\text{Var}(X|Y=3) = E(X^2|Y=3) - (E(X|Y=3))^2 = \frac{607}{79} - \left(\frac{213}{79}\right)^2 =$
(circle one) $\mathbf{\frac{1}{6241}}$ / $\mathbf{\frac{232}{6241}}$ / $\mathbf{\frac{247}{6241}}$ / $\mathbf{\frac{2584}{6241}}$

(k) *Show Var*$(X) = E[$*Var*$(X|Y)] +$ *Var*$(E(X|Y))$.
Since $E[\text{Var}(X|Y)] =$ (circle none, one or more)

i. $\sum_y \text{Var}(X|Y=y)P(Y=y) = \text{Var}(X|Y=1)P(Y=1) + \text{Var}(X|Y=2)P(Y=2) + \text{Var}(X|Y=3)P(Y=3)$
ii. $\left(\frac{4}{9}\right)(0.09) + \left(\frac{143}{144}\right)(0.12) + \left(\frac{2584}{6241}\right)(0.79)$
iii. 0.486255

and $\text{Var}(E(X|Y)) = E[(E[X|Y])^2] - (E[E(X|Y)])^2 = E[(E[X|Y])^2] - (E[X])^2$
where $E[(E[X|Y])^2] =$ (circle none, one or more)

i. $\sum_y (E[X|Y=y])^2 P(Y=y) = (E[X|Y=1])^2 P(Y=1) + (E[X|Y=2])^2 P(Y=2) + (E[X|Y=3])^2 P(Y=3)$
ii. $\left(\frac{24}{9}\right)^2(0.09) + \left(\frac{29}{12}\right)^2(0.12) + \left(\frac{213}{79}\right)^2(0.79)$
iii. 7.083744

and $(E[X])^2 =$ (circle none, one or more)

i. $\left(\sum_x xP(X=x)\right)^2 = ((1)P(X=1) + (2)P(X=2) + (3)P(X=3))^2$
ii. $((1)(0.11) + (2)(0.11) + (3)(0.78))^2$
iii. $2.67^2 = 7.1289$

And so $\text{Var}(E(X|Y)) = E[(E[X|Y])^2] - (E[X])^2 = 7.083744 - 7.1289 = -0.045156$
And so $\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E(X|Y)) = 0.486255 - 0.045156 = 0.441099$.

2. *Conditional Expectation, Continuous.* Consider the following distribution,

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0 < x < 1, 0 < y < 2 \\ 0 & elsewhere \end{cases}$$

(a) $f_X(x) =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty}\left(x^2 + \frac{xy}{3}\right)\,dy = \int_0^2\left(x^2 + \frac{xy}{3}\right)\,dy$

ii. $\left(yx^2 + \frac{xy^2}{6}\right)_0^2$

iii. $2x^2 + \frac{x2}{3}$

(b) $f_Y(y) =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3}\right)\, dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right)\, dx$

ii. $\left(\frac{1}{3}x^3 + \frac{x^2y^2}{6}\right)_0^1$

iii. $\frac{1}{3} + \frac{y^2}{6}$

(c) *Compute* $E[X|Y = 1]$.

Since $f_{X|Y}(x|y) =$ (circle none, one or more)

i. $\frac{f(x,y)}{f_Y(y)}$

ii. $\frac{x^2+\frac{xy}{3}}{\frac{1}{3}+\frac{y^2}{6}}$

iii. $\frac{6x^2+2xy}{2+y^2}$

then $E[X|Y] =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} x\left(\frac{6x^2+2xy}{2+y^2}\right)\, dx = \int_0^1 \frac{6x^3+2x^2y}{2+y^2}\, dx$

ii. $\left(\frac{\frac{6}{4}x^4+\frac{2}{3}x^3y}{2+y^2}\right)_{x=0}^1$

iii. $\frac{\frac{6}{4}+\frac{2}{3}y}{2+y^2} = \frac{9+4y}{12+6y^2}$

and so $E[X|Y = 1] = \frac{\frac{6}{4}+\frac{2}{3}(1)}{2+(1)^2} =$

(circle one) $\mathbf{\frac{12}{18}}$ / $\mathbf{\frac{13}{18}}$ / $\mathbf{\frac{14}{18}}$ / $\mathbf{\frac{15}{18}}$

(d) *Compute* $E[X|Y = 1.5]$.

$E[X|Y = 1.5] = \frac{\frac{6}{4}+\frac{2}{3}(1.5)}{2+(1.5)^2} =$

(circle one) $\mathbf{\frac{10}{17}}$ / $\mathbf{\frac{13}{17}}$ / $\mathbf{\frac{14}{17}}$ / $\mathbf{\frac{17}{17}}$

(e) *Compare* $E[E[X|Y]]$ *to* $E[X]$.

Since $E[E(X|Y)] =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} E(X|Y)f_Y(y)\, dy = \int_0^2 \left(\frac{9+4y}{12+6y^2}\right)\left(\frac{1}{3} + \frac{y^2}{6}\right)\, dy$

ii. $\int_0^2 \left(\frac{1}{4} + \frac{1}{9}y\right)\, dy$

iii. $\left(\frac{1}{4}y + \frac{1}{18}y^2\right)_{y=0}^2$

iv. $\frac{1}{4}(2) + \frac{1}{18}(2)^2 = \frac{13}{18}$

and $E[X] =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} xf_X(x)\, dx = \int_0^1 x\left(2x^2 + \frac{2x}{3}\right)\, dx$

ii. $\int_0^1 \left(2x^3 + \frac{2x^2}{3}\right)\, dx$

iii. $\left(\frac{1}{2}x^4 + \frac{2x^3}{9}\right)_{x=0}^{1}$

iv. $\frac{1}{2} + \frac{2}{9} = \frac{13}{18}$

In other words, $E[X]$ (circle one) **does / does not** equal $E[E(X|Y)]$.

(f) *Compute $E(X^2|Y=1)$.*
$E[X^2|Y] =$ (circle none, one or more)

i. $\int_{-\infty}^{\infty} x^2 \left(\frac{6x^2+2xy}{2+y^2}\right) dx = \int_0^1 \frac{6x^4+2x^3y}{2+y^2} dx$

ii. $\left(\frac{\frac{6}{5}x^5+\frac{2}{4}x^4y}{2+y^2}\right)_{x=0}^{1}$

iii. $\frac{\frac{6}{5}+\frac{2}{4}y}{2+y^2} = \frac{12+5y}{20+10y^2}$

and so $E[X|Y=1] = \frac{12+5(1)}{20+10(1)^2} =$
(circle one) $\mathbf{\frac{12}{30}}$ / $\mathbf{\frac{13}{30}}$ / $\mathbf{\frac{14}{30}}$ / $\mathbf{\frac{17}{30}}$

(g) *Compute Var$(X|Y=1)$.*
$\text{Var}(X|Y=1) = E(X^2|Y=1) - (E(X|Y=1))^2 = \frac{17}{30} - \left(\frac{13}{18}\right)^2 =$
(circle one) $\mathbf{\frac{1}{1620}}$ / $\mathbf{\frac{3}{1620}}$ / $\mathbf{\frac{4}{1620}}$ / $\mathbf{\frac{73}{1620}}$

(h) *Determine Var$(X|Y)$.*
$\text{Var}(X|Y) =$ (circle none, one or more)

i. $E(X^2|Y) - (E(X|Y))^2$

ii. $\frac{12+5y}{20+10y^2} - \left(\frac{9+4y}{12+6y^2}\right)^2$

3. *Prisoner's Escape and Three Doors.* A prisoner is faced with three doors. The first door leads to a tunnel that leads to freedom in 4 hours. The second door leads to a tunnel that returns the prisoner back to the prison in 5 hours. The third door leads to a tunnel that returns the prisoner back to the prison in 10 hours. Assume the prisoner is equally likely to choose any door. Let X represent the amount of time until the prisoner reaches freedom and let Y represent the door (1, 2 or 3) he chooses. What is the expected length of time until the prisoner reaches safety, $E[X]$?

(a) Since $E[X] = E[E(X|Y)] =$ (circle none, one or more)

i. $\sum_y E(X|Y=y)P(Y=y) = E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3)$

ii. $\frac{1}{3}[E(X|Y=1) + E(X|Y=2) + E(X|Y=3)]$

(b) and $E(X|Y=1) =$ (circle one)
(circle one) **4** / $\mathbf{5 + \boldsymbol{E[X]}}$ / $\mathbf{10 + \boldsymbol{E[X]}}$

(c) and $E(X|Y=2) =$ (circle one)
(circle one) **4** / $\mathbf{5 + \boldsymbol{E[X]}}$ / $\mathbf{10 + \boldsymbol{E[X]}}$

(d) and $E(X|Y=3) =$ (circle one)
(circle one) **4** / $\boldsymbol{5 + E[X]}$ / $\boldsymbol{10 + E[X]}$

(e) and so $E[X] = \frac{1}{3}[4 + (5 + E[X]) + (10 + E[X])]$ and so
$E[X] =$ (circle one) **16** / **17** / **19**

Review Chapter 8

Limit Theorems

8.1 Introduction

We look at various limit theorems used in probability theory, including some *laws of large numbers* and some *central limit theorems*.

8.2 Chebyshev's Inequality and the Weak Law of Large Numbers

We look at *Markov's inequality*, *Chebyshev's inequality* and the *weak law of large numbers*, which are given below.

- Markov's inequality: For nonnegative random variable X and for $a > 0$,
  $P\{X > a\} \leq \frac{E[X]}{a}$.
- Chebyshev's inequality: For random variable X with finite μ and σ^2 and for $k > 0$,
  $P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$.
- Weak law of large numbers: For a sequence of independent identical random variables X_i, each with finite $E[X_i] = \mu$ and for $\varepsilon > 0$,
  $P\left\{\left|\frac{X_1 + \cdots X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0$ as $n \to \infty$.

The first two inequalities allow us to specify (very loose) bounds on probabilities knowing only μ (Markov) or μ and σ (Chebyshev), when the distribution is not known. The first two inequalities also are used to prove further limit results, such as the third result, the weak law of large numbers.

Exercise 8.1 (Markov's Inequality and Chebyshev's Inequality)

1. *Markov's Inequality: Ph Levels In Soil.* Consider the following $n = 28$ Ph levels in soil samples taken at Sand Dunes Park, Indiana.

4.3	5	5.9	6.5	7.6	7.7	7.7	8.2	8.3	9.5
10.4	10.4	10.5	10.8	11.5	12	12	12.3	12.6	12.6
13	13.1	13.2	13.5	13.6	14.1	14.1	15.1		

Assume one of these twenty–eight samples is taken at random.

(a) **True** / **False** Markov's inequality can be rewritten in the following ways,

$$\begin{aligned} P\{X > a\} &\leq \frac{E[X]}{a} = \frac{\mu}{a} \\ P\{X < a\} &\geq 1 - \frac{E[X]}{a} \end{aligned}$$

(b) **True** / **False** The expected (or mean) Ph level is $\mu = 10.55$.
(Hint: Type the Ph levels into L_1, then STAT CALC 1:1–Var Stats.)

(c) If $a = 15$, Markov's inequality,

$$P\{X > 15\} \leq \frac{10.55}{15} \approx 0.703$$

allows us to say, *at most*, a proportion of 70.3% of the 28 Ph levels should be more than a Ph level of 15.
In fact, only 1 (one) of the 28 Ph levels, or $\frac{1}{28} = 0.036$ or 3.6%, are above 15 (look at the data above and check that only the Ph level 15.1 is above 15).
Markov's inequality (circle one) **has** / **has not** been violated in this case
Although not violated, Markov's inequality provides a (circle one) **good** / **bad** approximation to the proportion of Ph levels above 15.

(d) If $a = 11$, Markov's inequality,

$$P\{X > 11\} \leq \frac{10.55}{11} \approx$$

(circle one) **0.76** / **0.86** / **0.96**, allows us to say, at most, a proportion of 96% of the 28 Ph levels should be more than a Ph level of 11.
In fact, 14 of the 28 Ph levels, or $\frac{14}{28} = 0.50$ or 50%, are above a Ph level of 11.

(e) If $a = 7$, Markov's inequality,

$$P\{X > 7\} \leq \frac{10.55}{7} \approx$$

(circle one) **1.51** / **1.86** / **1.96**, allows us to say, at most, *all*[1] 28 Ph levels should be more than a Ph level of 7.
In fact, 24 of the 28 Ph levels, or $\frac{24}{28} = 0.86$ or 86%, are above a Ph level of 7.

[1]Even though $\frac{10.55}{7} \approx 1.51$, it is not possible to have more than 100% of all Ph levels have a Ph level more than 7.

(f) If $a = 11$, Markov's inequality,

$$P\{X < 11\} \geq 1 - \frac{10.55}{11} \approx$$

(circle one) **0.04** / **0.05** / **0.06**, allows us to say, *at least*, 4% of the 28 Ph levels should be *less* than a Ph level of 11.
In fact, 14 of the 28 Ph levels, or $\frac{14}{28} = 0.50$ or 50%, are less than a Ph level of 11.

2. *Chebyshev's Inequality: Ph Levels In Soil.* Consider the following $n = 28$ Ph levels in soil samples taken at Sand Dunes Park, Indiana.

4.3	5	5.9	6.5	7.6	7.7	7.7	8.2	8.3	9.5
10.4	10.4	10.5	10.8	11.5	12	12	12.3	12.6	12.6
13	13.1	13.2	13.5	13.6	14.1	14.1	15.1		

Assume one of these twenty–eight samples is taken at random.

(a) **True** / **False** Chebyshev's inequality can be rewritten in the following ways,

$$\begin{aligned}
P\{|X - \mu| \geq k\} &\leq \frac{\sigma^2}{k^2} \\
P\{|X - \mu| \geq \sigma k\} &\leq \frac{\sigma^2}{\sigma^2 k^2} = \frac{1}{k^2} \\
P\{|X - \mu| \leq \sigma k\} &\geq 1 - \frac{1}{k^2} \\
P\{\mu - \sigma k \leq X \leq \mu + \sigma k\} &\geq 1 - \frac{1}{k^2}
\end{aligned}$$

(b) **True** / **False** The expected Ph level and standard deviation in Ph level are $\mu = 10.55$ and $\sigma = 3.01$, respectively.
(Hint: Type the Ph levels into L_1, then STAT CALC 1:1–Var Stats.)

(c) The Ph level one standard deviation above the average is equal to $\mu + \sigma = 10.55 + 3.01 = 13.56$. The Ph level two standard deviations below the average is equal to $\mu - 2(3.01) = 4.53$. Determine (a), (b), (c), (d) and (e) in the figure and then fill in the table below.

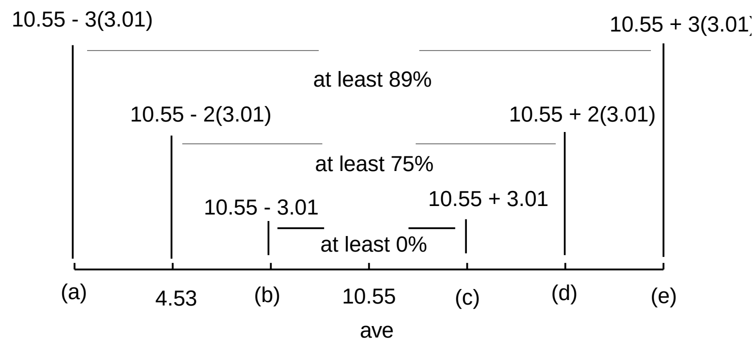


Figure 8.1 (Ph Levels 1, 2 and 3 Standard Deviations From Mean)

(a)	(b)	(c)	(d)	(e)

(d) The smallest Ph level, 4.3, is (circle one) **inside** / **outside** the interval between 7.54 and 13.56. Also, the Ph level, 10.5, is (circle one) **inside** / **outside** the interval (7.54, 13.56).

(e) Ph levels that are *within one standard deviation of the average*, refers to Ph levels that are (circle one) **inside** / **outside** the interval (7.54, 13.56). Ph levels that are within *two* standard deviations of the average, refers to Ph levels that are (circle one) **inside** / **outside** the interval (4.53, 16.57).

(f) Instead of saying "Ph levels that are within *one* standard deviation of the mean", it is also possible to say "Ph levels are within k standard deviations of the mean", where $k = 1$. If the Ph levels are within *two* standard deviations of the average,
then $k =$ **1** / **2** / **3**
If the Ph levels are within two and a half standard deviations of the average,
then $k =$ **1** / **1.5** / **2.5**

(g) If $k = 1.5$, then $1 - \frac{1}{k^2} = 1 - \frac{1}{1.5^2} \approx 0.56$ or 56%.
If $k = 2$, then $1 - \frac{1}{k^2} = \mathbf{\frac{1}{4}}$ / $\mathbf{\frac{2}{4}}$ / $\mathbf{\frac{3}{4}}$
which is equal to **25%** / **50%** / **75%**.

(h) Chebyshev's inequality,

$$P\{\mu - \sigma k \leq X \leq \mu + \sigma k\} \geq 1 - \frac{1}{k^2}$$
$$P\{10.55 - 2\sigma \leq X \leq 10.55 + 2\sigma\} \geq 1 - \frac{1}{2^2} = \frac{3}{4},$$

allows us to say, *at least* a $1 - \frac{1}{k^2} = 0.75$ proportion or 75% of the 28 Ph levels *should be* within *two* ($k = 2$) standard deviation of the average.
In fact, 27 of the 28 Ph levels (look at the data above and see for yourself),

or $\frac{27}{28} = 0.964$ or 96.4%, are in the interval (4.53, 16.57). Chebyshev's inequality (circle one) **has** / **has not** be violated in this case.

(i) Using Chebyshev's inequality, what proportion *should* fall within $k = 3$ standard deviations of the average?
$1 - \frac{1}{3^2} =$ (circle one) $\mathbf{\frac{3}{4}}$ / $\mathbf{\frac{6}{7}}$ / $\mathbf{\frac{8}{9}}$
In fact, what proportion of the Ph levels are actually in this interval (count the number in the interval (1.52,19.58))?
(circle one) $\mathbf{\frac{26}{28}}$ / $\mathbf{\frac{27}{28}}$ / $\mathbf{\frac{28}{28}}$

(j) Using Chebyshev's inequality, what proportion should fall within $k = 2.5$ standard deviations of the average?
$1 - \frac{1}{2.5^2} =$ (circle one) $\mathbf{\frac{20}{25}}$ / $\mathbf{\frac{21}{25}}$ / $\mathbf{\frac{22}{25}}$
In fact, what proportion of the Ph levels are actually in this interval (count the number in the interval (3.025,18.075))?
(circle one) $\mathbf{\frac{26}{28}}$ / $\mathbf{\frac{27}{28}}$ / $\mathbf{\frac{28}{28}}$

(k) Since *at least* 75% or 21 Ph levels are *inside* the interval (4.534, 16.574), then *at most*
(circle one) **25%** / **35%** / **45%**
of the levels are *outside* the interval (4.534, 16.574).

3. *Chebyshev and the Normal Distribution.* Let X be a normal random variable, with mean $\mu = 5$ and standard deviation $\sigma = 2$.

(a) Using your calculators, $P\{1 < X < 9\} =$ (circle one) **0.68** / **0.75** / **0.95**
(Hint: normalcdf(1,9,5,2))

(b) Using Chebyshev's inequality, $P\{1 < X < 9\} =$

$$\begin{aligned} P\{\mu - k\sigma \le X \le \mu + k\sigma\} &\ge 1 - \frac{1}{k^2} \\ P\{5 - 2(2) \le X \le 5 + 2(2)\} &\ge 1 - \frac{1}{2^2} = \end{aligned}$$

(circle one) **0.68** / **0.75** / **0.95**

(c) So, although Chebyshev's inequality is correct, it (circle one) **is** / **is not** a good approximation to the correct probability in this case.

4. *Weak Law of Large Numbers and Chebyshev.*

(a) **True** / **False** For a sequence of independent identical random variables X_i, each with finite $E[X_i] = \mu$ and so

$$E\left[\frac{X_1 + \cdots X_n}{n}\right] = \mu \quad \text{and} \quad \text{Var}\left[\frac{X_1 + \cdots X_n}{n}\right] = \frac{\sigma^2}{n}$$

then, from Chebyshev's inequality, $P\{|X-\mu| \geq \sigma k\} \leq \frac{\sigma^2}{\sigma^2 k^2}$,

$$P\left\{\left|\frac{X_1 + \cdots X_n}{n} - \mu\right| \geq \varepsilon\right\} \leq \frac{\sigma^2}{n\varepsilon^2}$$

which tends to zero, as $n \to 0$.

(b) Based on past experience, the mean test score is $\mu = 70$ and the variance in the test score is $\sigma^2 = 20$. How many students would have to take a test to be sure, that with probability of at least 0.85, the class average would be within $\varepsilon = 5$ of 70? Since

$$\begin{aligned} P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right\} &\leq \frac{\sigma^2}{n\varepsilon^2} \\ P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - 70\right| \geq 5\right\} &\leq \frac{20}{n5^2} \end{aligned}$$

and so $1 - 0.85 = \frac{20}{n5^2}$ when $n = \frac{20}{(0.15)25} \approx$ (circle one) **3** / **4** / **5**

(c) Based on past experience, the mean test score is $\mu = 50$ and the variance in the test score is $\sigma^2 = 15$. How many students would have to take a test to be sure, that with probability of at least 0.95, the class average would be within $\varepsilon = 7$ of 50?

$$n = \frac{15}{(0.05)7^2} \approx$$

(circle one) **3** / **4** / **6**

(d) Based on past experience, the mean test score is $\mu = 50$ and the variance in the test scores is $\sigma^2 = 15$. Determine the probability that the *average* score of 40 students will between 39 and 61. The *weak* law of large numbers,

$$P\left\{\left|\frac{X_1 + \cdots X_n}{n} - \mu\right| \leq \varepsilon\right\} \geq 1 - \frac{\sigma^2}{n\varepsilon^2},$$

can be used to approximate the probability In particular,

$$P\left\{\left|\frac{X_1 + \cdots X_n}{n} - 70\right| \leq 11\right\} \geq 1 - \frac{15^2}{(40)11^2} =$$

(circle one) **0.95** / **0.99** / **1**

8.3 The Central Limit Theorem

The *Central Limit Theorem* (CLT) says that as random sample size n increases, the probability distribution of the sum of independent identically distributed random variables, $\sum_{i=1}^{n} X_i$, tends to a normal distribution,

$$P\left\{\frac{X_1 + \cdots X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx, \quad n \to \infty$$

A related result is that

$$M_{Z_n}(t) \to M_X(t), \quad \text{implies} \quad F_{Z_n}(t) \to F_Z(t)$$

Exercise 8.2 (Central Limit Theorem)

1. *Using The Central Limit Theorem.*

   (a) *Sum.* Suppose X has a (*any!*) distribution where $\mu_X = 2.7$ and $\sigma_X = 0.64$. If $n = 35$, then determine $P\left(\sum_{i=1}^{35} X_i > 99\right)$.

      i. $\mu_{\sum X_i} = n\mu = 35(2.7) =$ (circle one) **93.5** / **94.5** / **95.5**.

      ii. $\sigma_{\sum X_i} = \sigma\sqrt{n} = 0.64\sqrt{35} =$ (circle one) **3.5** / **3.8** / **4.1**.

      iii. $P\left(\sum_{i=1}^{35} X_i > 99\right) \approx$ (circle one) **0.09** / **0.11** / **0.15**.
      (2nd DISTR normalcdf(99,E99,94.5,3.8))

   (b) *Another Sum.* Suppose X has a (*any!*) distribution where $\mu_X = -1.7$ and $\sigma_X = 1.6$. If $n = 43$, then determine $P\left(-76 < \sum_{i=1}^{43} X_i < -71\right)$.

      i. $\mu_{\sum X_i} = n\mu = 43(-1.7) =$ (circle one) **−73.5** / **−73.1** / **−72.9**.

      ii. $\sigma_{\sum X_i} = \sigma\sqrt{n} = 1.6\sqrt{43} =$ (circle one) **9.5** / **9.8** / **10.5**.

      iii. $P\left(-76 < \sum_{i=1}^{43} X_i < -71\right) \approx$ (circle one) **0.09** / **0.11** / **0.19**.
      (2nd DISTR normalcdf(-76,-71,-73.1,10.5))

   (c) *And Yet Another Sum.* Suppose X has a distribution where $\mu_X = 0.7$ and $\sigma_X = 1.1$. If $n = 51$, then
   $P\left(34.5 < \sum_{i=1}^{51} X_i < 35.1\right) \approx$ (circle one) **0.01** / **0.02** / **0.03**.

   (d) *Average.* Suppose X has a distribution where $\mu = 2.7$ and $\sigma = 0.64$. If $n = 35$, determine the chance the *average* (not sum!) is larger than 2.75, $P(\bar{X} > 2.75)$.

      i. $\mu_{\bar{X}} = \frac{n\mu}{n} = \mu =$ (circle one) **2.7** / **2.8** / **2.9**.

      ii. $\sigma_{\bar{X}} = \frac{\sigma\sqrt{n}}{n} = \frac{\sigma}{\sqrt{n}} = \frac{0.64}{\sqrt{35}} =$ (circle one) **0.11** / **0.12** / **0.13**.

      iii. $P\left(\bar{X} > 2.75\right) \approx$ (circle one) **0.30** / **0.32** / **0.35**.
      (2nd DISTR normalcdf(2.75,E99,2.7,0.11))

(e) *Another Average.* Suppose X has a distribution where $\mu_X = -1.7$ and $\sigma_X = 1.5$. If $n = 49$, then
$P(-2 < \bar{X} < 2.75) \approx$ (circle one) **0.58** / **0.58** / **0.92**.

(f) *Exponential Sum.* Suppose X has an *exponential* distribution where $\lambda = 4$ and where $n = 35$. Determine the chance of the sum of 35 independent identically distributed random variables, X, is greater than 9, $P\left(\sum_{i=1}^{35} X_i > 9\right)$ using the normal approximation.

i. $E[X] = \mu = \frac{1}{\lambda} =$ (circle one) **0.25** / **0.33** / **0.50**.
ii. $\mu_{\sum X_i} = n\mu = 35(0.25) =$ (circle one) **7.75** / **8.75** / **9.75**.
iii. $\text{Var}[X] = \sigma^2 = \frac{1}{\lambda^2} =$ (circle one) **0.0255** / **0.0335** / **0.0625**.
iv. $\sigma_{\sum X_i} = \sigma\sqrt{n} = \sqrt{0.0625}\sqrt{35} =$ (circle one) **0.0106** / **0.0335** / **1.48**.
v. Normal approximation,
$P\left(\sum_{i=1}^{35} X_i > 9\right) \approx$ (circle one) **0** / **0.11** / **0.43**.
(2nd DISTR normalcdf(9,E99,8.75,1.48))

(g) *Negative Binomial Sum.* Suppose X has a *negative binomial* distribution where the number of trials is $i = 500$, the required number of successes are $r = 4$ and chance of success on each trial is $p = 0.3$. Determine the chance of the sum of 35 independent identically distributed random variables, X, is greater than 450 successes, $P\left(\sum_{i=1}^{35} X_i > 450\right)$, using the normal approximation.

i. $E[X] = \mu = \frac{r}{p} =$ (circle one) **12.3** / **13.3** / **14.3**.
ii. $\mu_{\sum X_i} = n\mu = 35(13.3) =$ (circle one) **465.5** / **470.5** / **495.5**.
iii. $\text{Var}[X] = \sigma^2 = \frac{r(1-p)}{p^2} =$ (circle one) **29.3** / **31.1** / **34.3**.
iv. $\sigma_{\sum X_i} = \sigma\sqrt{n} = \sqrt{31.1}\sqrt{35} =$ (circle one) **33** / **35** / **37**.
v. Normal approximation,
$P\left(\sum_{i=1}^{35} X_i > 450\right) \approx$ (circle one) **0** / **0.68** / **0.75**.
(2nd DISTR normalcdf(450,E99,465.5,33))

(h) *Dice Sum.* What is the chance, in 30 rolls of a fair die, that the sum is between 100 and 105, $P\left(100 < \sum_{i=1}^{30} X_i < 105\right)$, using the normal approximation.

i. $E[X] = \mu = 1\left(\frac{1}{6}\right) + \cdots + 6\left(\frac{1}{6}\right) =$ (circle one) **2.3** / **3.5** / **4.3**.
ii. $\mu_{\sum X_i} = n\mu = 30(3.5) =$ (circle one) **100** / **105** / **110**.
iii. $E[X^2] = \mu = 1^2\left(\frac{1}{6}\right) + \cdots + 6^2\left(\frac{1}{6}\right) =$ (circle one) **2.3** / **3.5** / **15.2**.
iv. $\text{Var}[X] = E[X^2] - (E[X])^2 = 15.2 - 3.5^2$ (circle one) **2.9** / **3.1** / **3.3**.
v. $\sigma_{\sum X_i} = \sigma\sqrt{n} = \sqrt{2.9}\sqrt{30} =$ (circle one) **9.1** / **9.4** / **9.7**.
vi. Normal approximation,
$P\left(100 < \sum_{i=1}^{30} X_i < 105\right) \approx$ (circle one) **0** / **0.20** / **0.35**.
(2nd DISTR normalcdf(100,105,105,9.4))

2. *Understanding The Central Limit Theorem: Fishing in Montana.*

(a) The distributions of the average number of fish caught at a lake, $\bar{X}$, where $n = 1, 2, 3$ are given by

$x, n = 1$	1	2	3
$P(X = x)$	0.4	0.4	0.2

where $\mu_X = 1.8$ and $\sigma_X = 0.75$,

$\bar{x}, n = 2$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$P(\bar{X} = \bar{x})$	0.16	0.32	0.32	0.16	0.04

where $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{2}} = 0.53$,

$\bar{x}, n = 3$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3
$P(\bar{X} = \bar{x})$	0.064	0.192	0.288	0.256	0.144	0.048	0.008

where $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{3}} = 0.43$. The probability histograms of these three sampling distributions are given below.

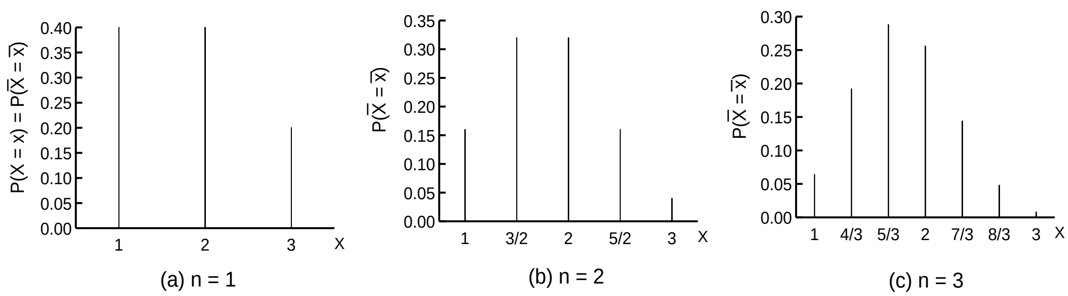


Figure 8.2 (Comparing Sampling Distributions Of Sample Mean)

As the random sample size, n, increases, the sampling distribution of the average, $\bar{X}$, changes shape and becomes more (circle one)

i. rectangular–shaped.
ii. bell–shaped.
iii. triangular–shaped.

In fact, the *central limit theorem* (CLT) says *no matter what the original distribution*, the sampling distribution of the average is typically normal when $n > 30$.

(b) Even though the sampling distribution becomes more normal–shaped as the random sample size increases, the mean of the average, $\mu_{\bar{X}} = 1.8$ (circle one)

i. decreases and is equal to $\frac{\sigma_X^2}{n}$,
ii. remains the same and is equal to $\mu_X = 1.8$,

iii. increases and is equal to $n\mu_X$,

and the standard deviation of the average, $\sigma_{\bar{X}}$ (circle one)

i. decreases and is equal to $\frac{\sigma_X}{\sqrt{n}}$.
ii. remains the same and is equal to σ_X.
iii. increases and is equal to $n\sigma_X$.

(c) After $n = 30$ trips to the lake, the distribution in the average number of fish caught is essentially normal (why?), where (circle one)

i. $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{3}} = 0.43$.
ii. $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{10}} = 0.24$.
iii. $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{30}} = 0.14$

(d) **True** / **False**. After $n = 30$ trips to the lake, the (approximate) chance the average number of fish caught is greater than 2.1 fish is given by (using your calculators)
$P(\bar{X} > 2.1) \approx 0.015$, where $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{30}} = 0.14$.
(2nd DISTR 2:normalcdf(2.1,E99,1.8,0.75/$\sqrt{30}$) ENTER)

(e) After 30 trips to the lake, the chance the average number of fish is *less than* 1.95 is $P(\bar{X} < 1.95) \approx$ (circle one) **0.73** / **0.86** / **0.94**.

(f) After $n = 35$ trips to the lake, the distribution in the average number of fish caught is essentially normal (why?), where (circle one)

i. $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{30}} = 0.14$.
ii. $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{35}} = 0.13$.
iii. $\mu_{\bar{X}} = 1.8$ and $\sigma_{\bar{X}} = \frac{0.75}{\sqrt{40}} = 0.12$

(g) After 35 trips to the lake, the chance the average number of fish is *less than* 1.95 is $P(\bar{X} < 1.95) \approx$ (circle one) **0.73** / **0.88** / **0.94**.

(h) After $n = 15$ trips to the lake, the distribution in the average number of fish caught (circle one) **is** / **is not** normal.

(i) The CLT is useful because (circle none, one or more):

i. No matter what the original distribution is, as long as a large enough random sample is taken, the average of this sample follows a normal (not a binomial or any other distribution) distribution.
ii. In practical situations where it is not known what probability distribution to use, as long as a large enough random sample is taken, the average of this sample follows a normal distribution.
iii. Rather than having to deal with many different probability distributions, as long as a large enough random sample is taken, the average of this sample follows *one* distribution, the normal distribution.

   iv. Many of the distributions in statistics rely in one way or another on the normal distribution because of the CLT.

(j) **True** / **False** The central limit theorem requires not only that $n \geq 30$, but also that a *random sample* of size $n \geq 30$ is used.