UCLA STAT 13

Introduction to Statistical Methods for the Life and Health Sciences

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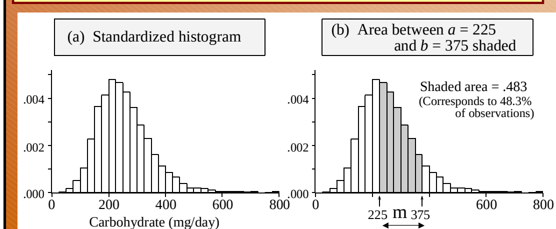
http://www.stat.ucla.edu/~dinov/courses_students.html

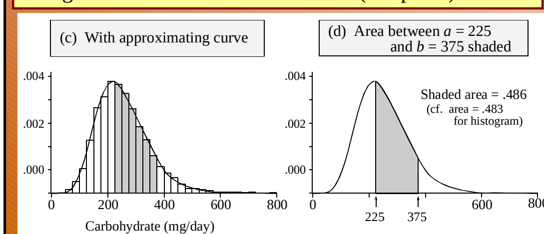
Chapter 6: Continuous Random Variables

- Continuous Random Variables
- The Normal Distribution
- Sums and differences of random quantities

Dietary intake of carbohydrate (mg/day) for N=5929 people from a variety of work environments. Standardized histogram plot is unimodal but skewed to the right (high values). Vertical scale is (relative freq.)/(interval width) = $f_j/(N^*m)$. The proportion of the data in [a : b] is the area under the standardized histogram on the range [a: b].

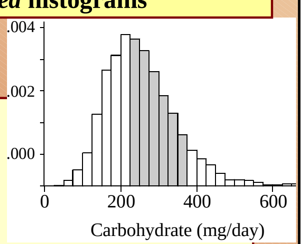


Superposition of a smooth curve (density function) on the standardized histogram (left panel). Area under the density curve on [a: b] = [225: 375] is analytically computed to be: 0.486 (right panel), which is close to the empirically obtained estimate of the area under the histogram on the same interval: 0.483 (left panel).



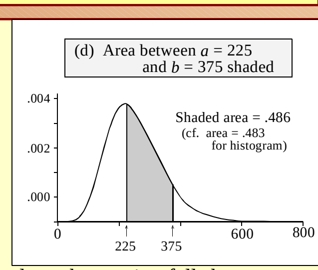
Standardized histograms

For a standardized histogram:



- The vertical scale is *Relative_frequency* / *Interval_width*
- ***Total area*** under histogram = 1
- ***Proportion*** of the data between *a* and *b* is the ***area*** under histogram between *a* and *b*

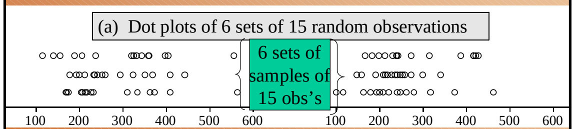
Probability and areas



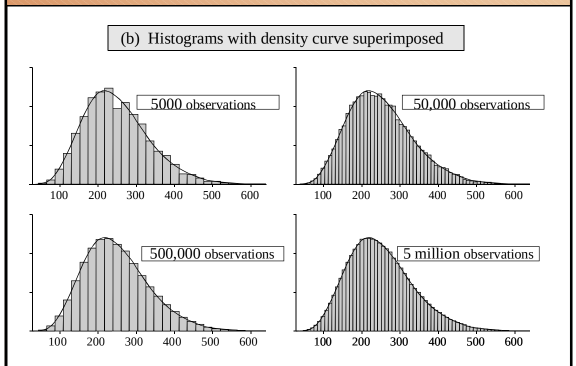
For a continuous *X*

- the probability a random observation falls between *a* and *b* = area under the density curve between *a* and *b*.

Sampling from the distribution in Fig. 6.1.1

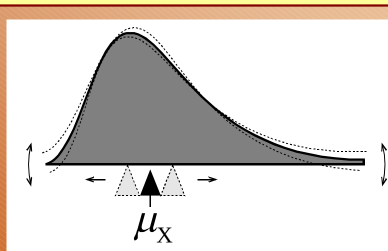


Note the fair amount of intra- and inter-group variability. What does that mean? Is that normal or expected?



From *Chance Encounters* by C.J. Wild and G.A.F. Seber, © John Wiley & Sons, 2000.

Visualizing the population mean



The population mean is the imaginary value of X where the density curve balances

Interval endpoints and continuous variables

Recall a continuous variable is one where the domain has no gaps in between the values the variable can take.

In calculations involving a continuous random variable we *do not have to worry* about whether interval endpoints are included or excluded.

Review

- How does a standardized histogram differ from a relative-frequency histogram? raw histogram? (f_j/mn)
- What graphic feature conveys the proportion of the data falling into a class interval for a standardized histogram? for a relative-frequency histogram? (area=width . height = m f_j/mn= f_j/n)
- What are the two fundamental ways in which random observations arise? (Natural phenomena, sampling experiments – choose a student at random and use the lottery method to record characteristics, scientific experiments - blood pressure measure)
- How does a density curve describe probabilities? (The probability that a random obs. falls in [a:b] is the area under the PDF on the same interval.)

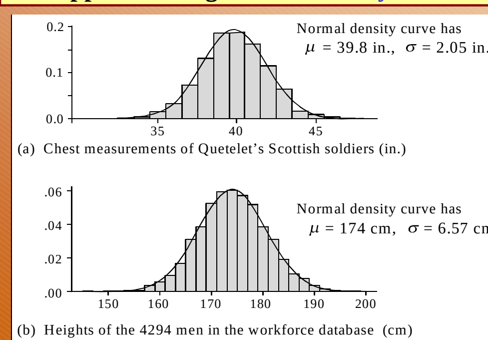
Review

- What is the total area under both a standardized histogram and a probability density curve? (1)
- When can histograms of data from a random process be relied on to closely resemble the density curve for that process? (large sample size, small histogram bin-size)
- What characteristic of the density curve does the mean correspond to? (imaginary value of *X*, where the density curve balances)

Review

- Does it matter whether <u>interval endpoints</u> are included or excluded when we calculate probabilities for a continuous random variable from the area? (No)
- Why? (Area[a:b] == Area(a:b))
- Are discrete variables the <u>same</u> or <u>different</u> in this regard, interval endpoint not effecting the area? (Different)

Two standardized histograms with approximating Normal density curve

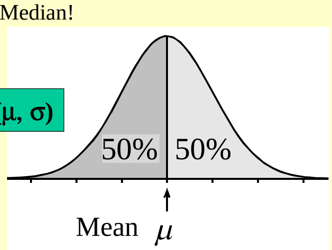


(a) Chest measurements of Quetelet's Scottish soldiers (in.)

(b) Heights of the 4294 men in the workforce database (cm)

The Normal distribution density curve

- Is symmetric about the mean! Bell-shaped and unimodal.
- Mean = Median!

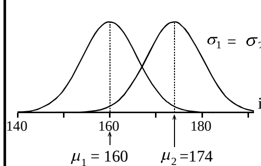


Effects of μ and σ

(a) Changing μ

shifts the curve along the axis

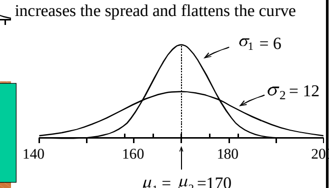
Mean is a measure of …
central tendency



(b) Increasing σ

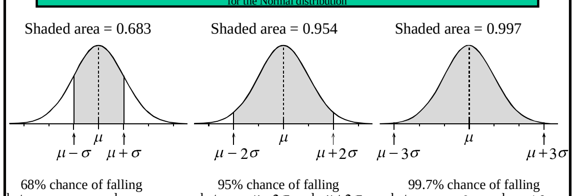
increases the spread and flattens the curve

Standard deviation is a measure of …
variability/spread



Understanding the standard deviation: σ

Probabilities/areas and numbers of standard deviations
for the Normal distribution

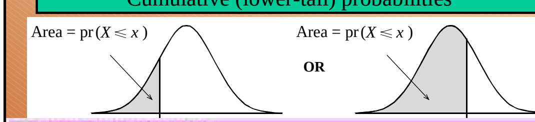


68% chance of falling between $\mu-\sigma$ and $\mu+\sigma$

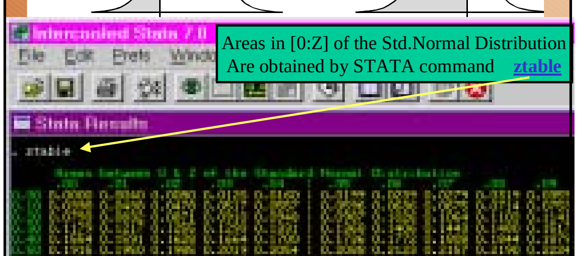
95% chance of falling between $\mu-2\sigma$ and $\mu+2\sigma$

99.7% chance of falling between $\mu-3\sigma$ and $\mu+3\sigma$

Probabilities supplied by computer programs – Cumulative (lower-tail) probabilities

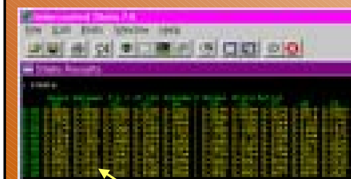


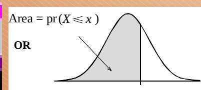
Areas in [0:Z] of the Std.Normal Distribution
Are obtained by STATA command ztable



Probabilities supplied by computer programs – Cumulative (lower-tail) probabilities

Problem: To find P(X<=180), when μ=174 and σ=6.57
Convert to Standard units: Y = (X- μ)/ σ = 6/6.57 = 0.91
Look-up the Normal Distribution Table: 0.3186
Final cumulative (lower-tail) result: 0.5+ 0.3186= 0.819





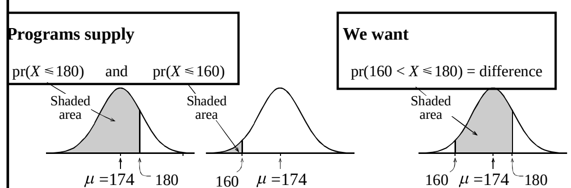
Finding Help with STATA

- http://www.stat.ucla.edu/~dinov/courses_students.dir

Basic method for obtaining probabilities

- Sketch a Normal curve, marking the mean and other values of interest.
- Shade the area under the curve that gives the desired probability.
- Devise a way of getting the desired area from lower-tail areas.
- Obtain component lower-tail probabilities from a computer program

(a) Computing pr(160 < $X \leq$ 180)



pr(160 < $X \leq$ 180) = pr($X \leq$180) − pr($X \leq$160)

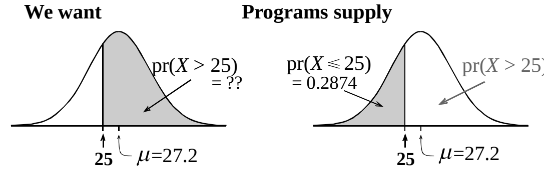
Fig. 6.2.4(c)

(c) More Normal probabilities
(values obtained from *Minitab*)

b	pr($X \leq$ b)	a	pr($X \leq$ a)	pr(a < $X \leq$ b) = difference
167.6	0.165	152.4	0.001	0.164
177.8	0.718	167.6	0.165	0.553
177.8	0.718	152.4	0.001	0.717
182.9	0.912	167.6	0.165	0.747

Note: 152.4cm = 5ft, 167.6cm = 5ft 6in., 177.8cm = 5ft 10in., 182.9cm = 6ft

Obtaining an upper-tail probability



Since total area under curve = 1, pr(X > 25) = 1 - pr($X \leq$ 25)

Generally, **pr(X > x) = 1 − pr($X \leq x$)**

Review

- What features of the Normal curve do μ and σ visually correspond to? (point-of-balance; width/spread)
- What is the probability that a random observation from a normal distribution is smaller than the mean? (0.5) larger than the mean? (0.5) exactly equal to the mean? (0.0) Why?

Review

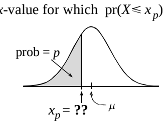
- Approximately, what is the probability that a random observation from a normal distribution falls within 1 standard deviation (SD) of the mean? (0.68) 2 SD's? (0.95) 3 SD's? (0.997)
- Computer programs may provide cumulative or partial probabilities for the normal distribution. What is a the difference between these? Can we get one from the other?

The inverse problem – Percentiles/quantiles

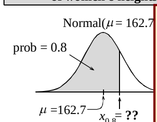
(a) *p*-Quantile

Programs supply x_p

x-value for which $pr(X \leq x_p) = p$



(b) 80th percentile (0.8-quantile) **of women's heights**



Program returns
Thus 80% lie belo

(c) Further percentiles of women's heights

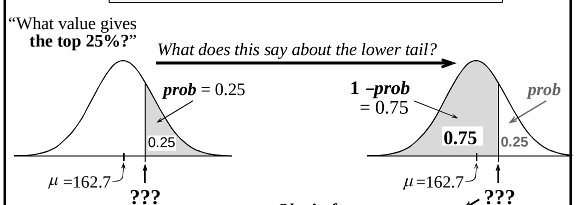
Percent	1%	5%	10%	20%	30%	70%	80%	90%	95%
Propn	0.01	0.05	0.1	0.2	0.3	0.7	0.8	0.9	0.95
Percentile	*(or quantile)*								
(cm)	148.3	152.5	154.8	157.5	159.4	166.0	167.9	170.6	172.9
(ft'in")	4'10"	5'0"	5'0"	5'2"	5'2"	5'5"	5'6"	5'7"	5'8"

80% of people have height below the 80th percentile. This is EQ to saying there's 80% chance that a random observation from the distribution will fall below the 80th percentile.

The inverse problem is what is the height for the 80th percentile/quantile? So far we studied given the height value what's the corresponding percentile?

The inverse problem – *upper-tail* percentiles/quantiles

Obtaining an inverse upper-tail probability



[Program returns 166.88]

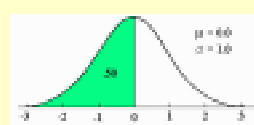
Review

- What is meant by the 60th percentile of heights?
- What is the difference between a percentile and a quantile? (percentile used in expressing results in %, whereas quantiles used to express results in term of probabilities)
- The lower quartile, median and upper quartile of a distribution correspond to special percentiles. What are they? express in terms of quantiles. (25%, 50%, 75%)
- Quantiles are sometimes called inverse cumulative probabilities. Why?

Standard Normal Curve

- The standard normal curve is described by the equation:

$$y = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

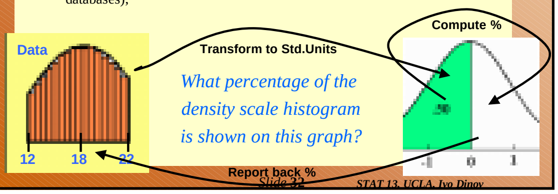


Where remember, the natural number e ~ 2.7182…

We say: X~Normal(μ, σ), or simply X~N(μ, σ)

Standard Normal Approximation

- The standard normal curve can be used to estimate the percentage of entries in an interval for any process. Here is the protocol for this approximation:
  - Convert the interval (we need the assess the percentage of entries in) to standard units. We saw the algorithm already.
  - Find the corresponding area under the normal curve (from tables or online databases);

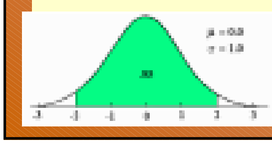


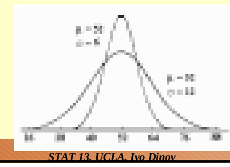
General Normal Curve

- The general normal curve is defined by:
  - Where μ is the average of (the symmetric) normal curve, and σ is the standard deviation (spread of the distribution).

$$y = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

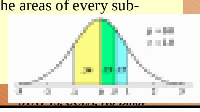
  - Why worry about a standard and general normal curves?
  - How to convert between the two curves?





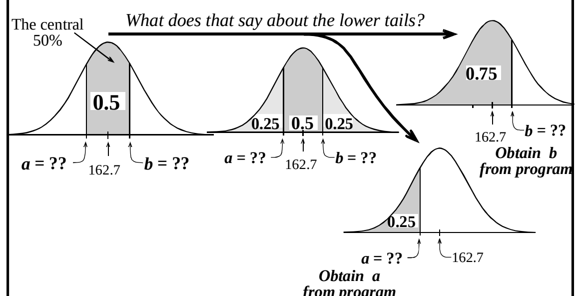
Areas under Standard Normal Curve – Normal Approximation

- Protocol:
  - Convert the interval (we need to assess the percentage of entries in) to Standard units. Actually convert the end points in Standard units.
    - In general, the transformation X → (X-μ)/σ, standardizes the observed value X, where μ and σ are the average and the standard deviation of the distribution X is drawn from.
  - Find the corresponding area under the normal curve (from tables or online databases);
    - Sketch the normal curve and shade the area of interest
    - Separate your area into individually computable sections
    - Check the Normal Table and extract the areas of every sub-section
    - Add/compute the areas of all sub-sections to get the total area.



Obtaining central range for symmetric distributions

What values contain the **central** 50%?



The z-score

- The ***z*-score** of *x* is the number of standard deviations *x* is from the mean. (Body-Mass-Index, BMI)

TABLE 6.3.1 **Examples of *z* -Scores**

x	*z* -score = (*x* - *μ*)/*σ*		Interpretation
Male BMI values (kg/m²)			
25	(25-27.3)/4.1 =	-0.56	25 kg/m² is 0.56 sd's below the mean
35	(35-27.3)/4.1 =	1.88	35 kg/m² is 1.88 sd's above the mean
Female heights (cm)			
155	(155-162.7)/6.2 =	-1.24	155cm is 1.24 sd's below the mean
180	(180-162.7)/6.2 =	2.79	180cm is 2.79 sd's above the mean

Male BMI-values: μ=27.3, σ=4.1 Females heights: μ=162.7, σ=6.2

- Which ones of these are unusually large/small/away from the mean?

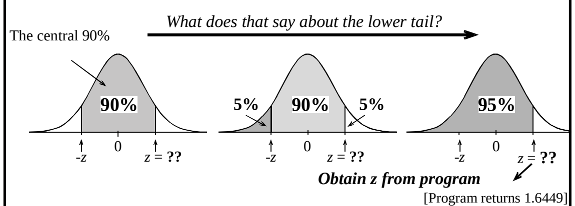
The standard Normal distribution

Standard Normal distribution:

mean(μ) = 0, SD(σ)= 1

Working in standard units

What values contain the **central** 90%?



- The **z-score** of x is the number of standard deviations x is away from the mean.

Working in standard units (# of SD's)

TABLE 6.3.2 Central Ranges

		Male BMI values		Female heights	
Percentage	**z**	$\mu - z\sigma$	$\mu + z\sigma$	$\mu - z\sigma$	$\mu + z\sigma$
80%	1.2816	22.05	32.55	154.8	170.6
90%	1.6449	20.56	34.04	152.5	172.9
95%	1.9600	19.26	35.34	150.5	174.9
99%	2.5758	16.74	37.86	146.7	178.7
99.9%	3.2905	13.81	40.79	142.3	183.1

Male BMI-values: μ=27.3, σ=4.1
Females heights: μ=162.7, σ=6.2

Standardizing $Z = (X - \mu)/\sigma$
Inverting $X = Z\sigma + \mu$

TABLE 6.3.3 Using z-score tables

As an example, we shall find pr($Z \le$ 1.1357) using part of the table given in Appendix A4 (reproduced below).

Step 1: Correct the z-value to two decimal places, that is, use z = 1.14.
Step 2: Look down the z column until you find 1.1. This tells you which row to look in.
Step 3: The second decimal place, here 4, tells you which column to look in.
Step 4: The entry in the table corresponding to that row and column is pr($Z \le$ 1.14) = 0.873

z	0	1	2	3	4	5	6	7	8	9
1.0	.841	.844	.846	.848	.851	.853	.855	.858	.860	.862
1.1	.864	.867	.869	.871	.873	.875	.877	.879	.881	.883
1.2	.885	.887	.889	.891	.893	.894	.896	.898	.900	.901
1.3	.903	.905	.907	.908	.910	.911	.913	.915	.916	.918
1.4	.919	.921	.922	.924	.925	.926	.928	.929	.931	.932

Quincunx

Continuous Variables and Density Curves

- There are no gaps between the values a continuous random variable can take.
- Random observations arise in two main ways: (i) by sampling populations; and (ii) by observing processes.

The density curve

- The probability distribution of a continuous variable is represented by a density curve.
  - ***Probabilities*** are represented by ***areas under the curve***,
    - the probability that a random observation falls between a and b equal to the area under the density curve between a and b.
  - The total area under the curve equals 1.
  - The population (or distribution) mean μ_X = E(X), is where the density curve balances.
  - When we calculate probabilities for a continuous random variable, it does not matter whether interval endpoints are included or excluded.

For any random variable X

- $E(aX + b) = a\,E(X) + b$ and $SD(aX + b) = |\,a\,|\,SD(X)$

The Normal distribution

$X \sim \text{Normal}(\mu_x = \mu, \sigma_x = \sigma)$

Features of the Normal density curve:

- The curve is a symmetric bell-shape centered at μ.
- The standard deviation σ governs the spread.
  - 68.3% of the probability lies within 1 standard deviation of the mean
  - 95.4% within 2 standard deviations
  - 99.7% within 3 standard deviations

Probabilities

- Computer programs provide lower-tail (or cumulative) probabilities of the form $\text{pr}(X \le x)$
  - We give the program the x-value; it gives us the probability.
- Computer programs also provide inverse lower-tail probabilities (or quantiles)
  - We give the program the probability; it gives us the x-value.
- When calculating probabilities, we shade the desired area under the curve and then devise a way of obtaining it via lower-tail probabilities.

Standard Units

The *z-score* of a value *a* is ….

- the number of standard deviations a is away from the mean
- positive if a is above the mean and negative if a is below the mean.

The ***standard Normal*** distribution has $\mu = 0$ and $\sigma = 0$.

- We usually use Z to represent a random variable with a standard Normal distribution.

Ranges, extremes and *z*-scores

Central ranges:

- $P(-z \le Z \le z)$ is the same as the probability that a random observation from an arbitrary Normal distribution falls within z SD's either side of the mean.

Extremes:

- $P(Z \ge z)$ is the same as the probability that a random observation from an arbitrary Normal distribution falls more than z standard deviations above the mean.
- $P(Z \le -z)$ is the same as the probability that a random observation from an arbitrary Normal distribution falls more than z standard deviations below the mean.

Combining Random Quantities

Variation and independence:

- No two animals, organisms, natural or man-made objects are ever identical.
- There is always variation. The only question is whether it is large enough to have a practical impact on what you are trying to achieve.
- Variation in component parts leads to even greater variation in the whole.

Independence

We model variables as being independent ….

- if we think they relate to physically independent processes
- and if we have no data that suggests they are related.

Both sums and differences of independent random variables are more variable than any of the component random variables

Formulas

- For a constant number a, $E(aX) = aE(X)$ and $SD(aX) = |a|\, SD(X)$.
- Means of sums and differences of random variables act in an obvious way
  - the mean of the sum is the sum of the means
  - the mean of the difference is the difference in the means
- For independent random variables, (cf. Pythagorean theorem),

$$SD(X_1 + X_2) = SD(X_1 - X_2) = \sqrt{SD(X_1)^2 + SD(X_2)^2}$$

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

[ASIDE: Sums and differences of independent Normally distributed random variables are also Normally distributed]

Example

- Assumption: Crime rate for individuals is independent of family relations!
- Let X = RV representing the number of crimes an average individual commits in a 5 yr span.
- Let X1 and X2 be the crime rates of a husband and the wife in one family. What is the expected crime rate for this family given that E(X) = 1.4 and SD(X) = 0.7?

$$SD(X_1 + X_2) = SD(X_1 - X_2) = \sqrt{SD(X_1)^2 + SD(X_2)^2}$$

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

Example

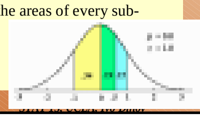
- Assumption: Crime rate for individuals is independent of family relations!
- If X1+X2 = 3X-1, dependent case
- X1 and X2 are independent (family relation)
- Suppose X1+X2 = 5, is this atypical?

$$SD(X_1 + X_2) = SD(X_1 - X_2) = \sqrt{SD(X_1)^2 + SD(X_2)^2}$$

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

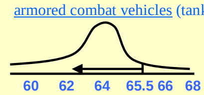
Areas under Standard Normal Curve – Normal Approximation

- Protocol:
  - Convert the interval (we need to assess the percentage of entries in) to Standard units. Actually convert the end points in Standard units.
    - In general, the transformation X → (X-μ)/σ, standardizes the observed value X, where μ and σ are the average and standard deviation of the distribution X is drawn from.
  - Find the corresponding area under the normal curve (from tables or online databases);
    - Sketch the normal curve and shade the area of interest
    - Separate your area into individually computable sections
    - Check the Normal Table and extract the areas of every sub-section
    - Add/compute the areas of all sub-sections to get the total area.



Areas under Standard Normal Curve – Normal Approximation, Scottish Army Recruits

- The mean height is 64 *in* and the standard deviation is 2 *in*.
  - Only recruits shorter than 65.5 in will be trained for tank operation. What percentage of the incoming recruits will be trained to operate armored combat vehicles (tanks)?

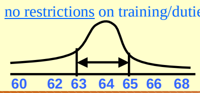


X → (X-64)/2

65.5 → (65.5-64)/2 = ¾

Percentage is 77.34%

  - Recruits within ½ standard deviations of the mean will have no restrictions on duties. About what percentage of the recruits will have no restrictions on training/duties?



X → (X-64)/2

65 → (65-64)/2 = ½

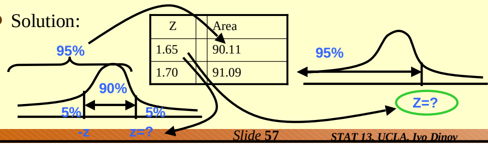
63 → (63-64)/2 = -½

Percentage is 38.30%

Percentiles for Standard Normal Curve

- When the histogram of the observed process follows the normal curve Normal Tables (of any type, as described before) may be used to estimate percentiles. The N-th percentile of a distribution is P is N% of the population observations are less than or equal to P.
- Example, suppose the Math-part SAT scores of newly admitted freshmen at UCLA averaged 535 (out of [200:800]) and the SD was 100. Estimate the 95 percentile for the score distribution.
- Solution:

Z	Area
1.65	90.11
1.70	91.09



Percentiles for Standard Normal Curve

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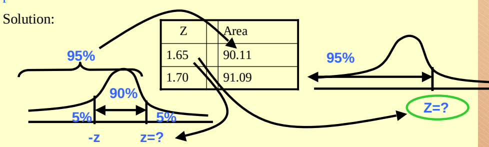


- Z=1.65 (std. Units) → 700 (data units), since

  $X \to (X - \mu)/\sigma$, converts data to standard units and

  $X \to \sigma X + \mu$, converts standard to data units!

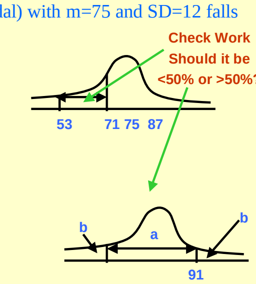
  $\sigma = 100$; $\mu = 535$, $100 \times 1.65 + 535 = 700$.

Summary

1. The Standard Normal curve is symmetric w.r.t. the origin (0,0) and the total area under the curve is 100% (1 unit)
2. Std units indicate how many SD's is a value below (-)/above (+) the mean
3. Many histograms have roughly the shape of the normal curve (bell-shape)
4. If a list of numbers follows the normal curve the percentage of entries falling within each interval is estimated by: 1. Converting the interval to StdUnits and, 2. Computing the corresponding area under the normal curve (Normal approximation)
5. A histogram which follows the normal curve may be reconstructed just from (μ,σ^2), mean and variance=std_dev²
6. Any histogram can be summarized using percentiles
7. $E(aX+b)=aE(X)+b$, $Var(aX+b)=a^2Var(X)$, where E(Y) the the mean of Y and Var(Y) is the square of the StdDev(Y),

Example – work out in your notebooks

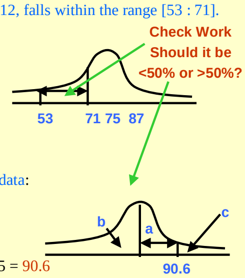
1. Compute the chance a random observation from a distribution (symmetric, bell-shaped, unimodal) with m=75 and SD=12 falls within the range [53 : 71].



Example – work out in your notebooks

Compute the chance a random observation from a distribution (symmetric, bellshaped & unimodal) with mean μ=75 and SD σ=12, falls within the range [53 : 71].

1. (53-75)/12 = -11/6=-1.83 Std unit
2. (71-75)/12=-0.333(3) Std units
3. Area[53:71] =
4. (SN_area[-1.83 : 0] – SN_area[-0.33 : 0])
5. = (0.4664 - 0.1293) = 0.34 (34%)
6. Compute the 90th percentile for the same data:
7. b+a+c=100% a=40%
8. b=50% c=10% Z=1.28 SU
9. 90% Percentile = $\sigma 1.3 + \mu$ =12x1.28+75 = 90.6

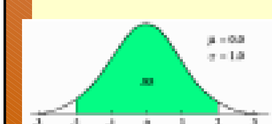


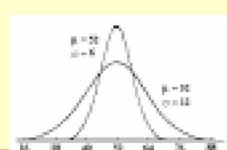
General Normal Curve

- The general normal curve is defined by:
  - Where μ is the average of (the symmetric) normal curve, and σ is the standard deviation (spread of the distribution).

$$y = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

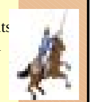
  - Why worry about a standard and general normal curves?
  - How to convert between the two curves?





Areas under Standard Normal Curve

- Many histograms are similar in shape to the standard normal curve. For example, persons height. The height of all incoming female army recruits is measured for custom training and assignment purposes (e.g., very tall people are inappropriate for constricted space positions, and very short people may be disadvantages in certain other situations). The mean height is computed to be 64 *in* and the standard deviation is 2 *in*. Only recruits shorter than 65.5 in will be trained for tank operation and recruits within ½ standard deviations of the mean will have no restrictions on duties.
  - What percentage of the incoming recruits will be trained to operate armored combat vehicles (tanks)?
  - About what percentage of the recruits will have no restrictions on training/duties?

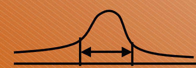


Standard Normal Curve – Table differences

- There are different tables and computer packages for representing the area under the standard normal curve. But the results are always interchangeable.

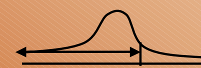
Area under Normal curve on [-z : z]

Z	Area
0.50	38.29
1.0	68.27



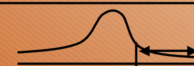
Area under Normal curve on [-infinity : z]

Z	Area
0.50	69.15
1.0	84.13



Area under Normal curve on [z: infinity]

Z	Area
0.50	30.85
1.0	15.87



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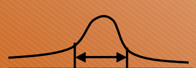

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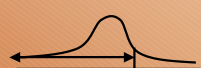

Area under Normal curve on [0 : z]

Z	Area
0.50	30.85
1.0	15.87