Stats 110B
HW3 Suggested Solutions
http://www.stat.ucla.edu/~dinov/courses_students.dir/03/Spr/Stat110B.dir/STAT110B.html
http://www.stat.ucla.edu/~dinov/courses_students.dir/03/Spr/Stat110B.dir/assignments.html

1. (a) $f_X(x) = \mathbf{c}\, x^2 e^{-\gamma x}$
$K = mX^2/2$
$X = (2K/m)^{1/2}$
$F_K(k) = P(K \le k) = P((mX^2/2) \le k)$
$= P(X \le (2k/m)^{1/2})$
$= \int_0^{(2k/m)^{\wedge}0.5} \mathbf{c}\, x^2 e^{-\gamma x}\, dx$

Therefore $f_K(k)$ = derivative of $\int_0^{(2k/m)^{\wedge}0.5} \mathbf{c}\, x^2 e^{-\gamma x}\, dx$
$= 2^{1/2} m^{-3/2} k^{1/2} c\, e^{-\gamma(2k/m)^{\wedge}0.5}$

Jacobian transformation approach :
If we have a function $x = v(k)$
Then $f(k) = |v'(k)|\, f(v(k))$
Here $v(x) = x = (2k/m)^{1/2}$
$v'(x) = (2/m)^{1/2} 1/2(k^{-1/2})$
So $f_K(k) = (2/m)^{1/2} 1/2(k^{-1/2})\, \mathbf{c}\, 2k/m\, e^{-\gamma(2k/m)^{\wedge}0.5}$
$= 2^{1/2} m^{-3/2} k^{1/2} c\, e^{-\gamma(2k/m)^{\wedge}0.5}$

(b) $f_K(k) = 2^{1/2} m^{-3/2} k^{1/2} c\, e^{-\gamma(2k/m)^{\wedge}0.5}$
$E(k) = \int_0^{\infty} k\, 2^{1/2} m^{-3/2} k^{1/2} c\, e^{-\gamma(2k/m)^{\wedge}0.5}\, dk$
$= \int_0^{\infty} 2^{1/2} m^{-3/2} k^{3/2} c\, e^{-\gamma(2k/m)^{\wedge}0.5}\, dk$
Now let $t = (2k/m)^{1/2}$
$k = (mt^2)/2$
$dk = mt\, dt$
expected value $= \int_0^{\infty} 2^{1/2} m^{-3/2} k^{3/2} c\, e^{-\gamma(2k/m)^{\wedge}0.5}\, dk$
$= \int_0^{\infty} 2^{1/2} m^{-3/2} ((mt^2)/2)^{3/2} c\, e^{-\gamma t}\, mt\, dt$
$= \int_0^{\infty} cm/2\, t^4 e^{-\gamma t}\, dt$
$= 3cm(1/\gamma^4) \int_0^{\infty} \gamma^4/6\, t^4 e^{-\gamma t}\, dt$
$\int_0^{\infty} \gamma^4/6\, t^4 e^{-\gamma t}\, dt$ = expectation of gamma distribution with $\alpha = 4$ and $\beta = 1/\gamma$
$= \alpha\, \beta = 4/\gamma$
expectation of $k = 3cm(4/\gamma) = 12cm/\gamma$

2. $f(x;y)$ should be $(9/26)(xy+y)^2$ when $0 \le x \le 2;\ 0 \le y \le 1$
(a) $f(y|x) = f(x;y)/f_x(x)$
$f_x(x) = \int_0^1 (9/26)(xy+y)^2 dy = 9/26 \times 1/3(xy+y)^3(1/(x+1)|_0^1$
$= 3/26(x+1)^2$
So $f(y|x) = ((9/26)(xy+y)^2)/(3/26(x+1)^2)$
$= 3y^2$
(b) $P(Y|X<1/2) = (\int_0^{0.5} f(x;y))/(\int_0^{0.5} f_x(x))$
$= (\int_0^{0.5}(9/26)(xy+y)^2 dx) / (\int_0^{0.5} 3/26(x+1)^2 dx)$
$= (3/26\, y^2(x+1)^3|_0^{0.5}) / (1/26\, (x+1)^3|_0^{0.5})$

$= 3y^2$

$P(Y<1/2 \mid X<1/2) = \int_0^{0.5} 3y^2\, dy = y^3 \mid_0^{0.5} = 0.125$

(c) $E(Y \mid X = x)$

$= (\int_{-\infty}^{\infty} y\, f(y|x)\, dy = \int_0^1 y\, 3y^2\, dy = 3/4\, y^4|_0^1 = 3/4$

3. possible p.d.f. :
   Binomial
   Poisson
   Geometric
   Hypergeometric
   Negative Binomial
   Uniform
   Normal
   Weibull
   Gamma
   Exponential
   Log
   Chi-squared
   F (etc.)

(d) If we have a sample $X_1 \ldots X_n$ of size n from a distribution with mean μ and variance σ^2 and if n is sufficiently(say greater than 30), then the sample mean X-bar would be approximately normally distributed with mean μ and variance σ^2/n, regardless of the distribution of X's. Therefore when the sample size is large, we do not have to care about the parent distribution to estimate X-bar, this makes the calculations a lot easier.

4. sample mean(x-bar) $= \Sigma x_i p_i$

$E(\text{x-bar}) = E(\Sigma x_i p_i)$

$= \Sigma p_i E(xi)$

$= \Sigma p_i \mu$

$= \mu$ (unbiased)

5. sample mean = 87.67; $\Sigma(\text{x-bar} - x)^2 = 780.67$; n =12

95% CI = $(780.67/\chi^2_{0.025,\, 11}\, ,\, 780.67/\chi^2_{0.975,\, 11})$

= (780.67/21.92 , 780.67/ 3.816)

= (35.61, 204.58)

If we have a wide interval, that means we have a lot of variability in the scores.