UCLA STAT 110B
Applied Statistics for Engineering and the Sciences

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http://www.stat.ucla.edu/~dinov/courses_students.html

Analysis of Variance - ANOVA

Use to analyze data -

1. That involves sampling from more than two populations, or
2. From experiments in which more than two treatments have been used

Use to compare more than two treatment or population means

Definitions

Factor – The characteristic that distinguishes the treatments or populations from one another

Levels – This refers to the different treatments or populations

Single-Factor ANOVA (chapter 10)

Multi-Factor ANOVA (chapter 11)

Example

An experiment to study the effects of four different brands of gasoline (Exxon, Conoco, Shell, Texaco) on the fuel efficiency (mpg) of a car

- Factor – Gasoline Brand
- Levels – the 4 brands (Exxon, Conoco, Shell, Texaco)
- Single-Factor ANOVA

Example

An experiment to study the effects of four different brands (Exxon, Conoco, Shell, Texaco) and three different types of gasoline (regular, midgrade, premium) on the fuel efficiency (mpg) of a car

- Factor – Gasoline Brand, Gasoline Type
- Levels – the 4 brands (Exxon, Conoco, Shell, Texaco), the 3 types (Regular, midgrade, premium)
- Two-Factor ANOVA

Mathematical Specification – 1 Way ANOVA

I = Number of Populations or Treatments being Compared

μ_i =The mean of population ***i*** or the true average when treatment ***i*** is applied

The hypotheses of interest are:

$H_o: \mu_1 = \mu_2 = \ldots\ldots = \mu_i$

H_a: at least two of the μ_i's are different

Single-Factor ANOVA

J = Number of observations in each sample; Assume each sample has same # observations

$X_{i,j}$ = j^{th} measurement from the ith population or treatment

A dot indicates that we have summed over that subscript

$$X_{i\cdot} = \sum_{j=1}^{J} X_{ij} \qquad X_{\cdot j} = \sum_{i=1}^{I} X_{ij}$$

Single-Factor Cont'd

Individual Sample Means:

$$\overline{X}_{i\cdot} = \frac{\sum_{j=1}^{J} X_{ij}}{J}, \; i = 1,\ldots,I$$

Single-Factor Cont'd

Grand Mean:

$$\overline{X}_{\cdot\cdot} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij}}{IJ}$$

Example: Gasoline

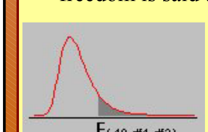
Assumptions

1. The ***I*** population or treatment distributions are each Normal
2. Each of these distributions has (approximately) the same variance, i.e. $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2 = \sigma^2$

$$X_{ij} \sim N(\mu_i, \sigma^2)$$

F-distribution

- F-distribution is the ratio of two χ^2 random variables.
- Snedecor's F distribution is most commonly used in tests of variance (e.g., ANOVA). The ratio of two chi-squares divided by their respective degrees of freedom is said to follow an F distribution



$$SD^2(Y) = \frac{1}{N-1}\sum_{k=1}^{N}(Y_k - \overline{Y})^2; \; SD^2(X) = \frac{1}{M-1}\sum_{l=1}^{M}(X_l - \overline{X})^2$$

$$W_Y = \frac{N-1}{\sigma_Y^2} SD^2(Y); \; W_X = \frac{M-1}{\sigma_X^2} SD^2(X);$$

$$F_o = \frac{W_Y/(N-1)}{W_X/(M-1)} \sim F(df_1 = N-1, df_2 = M-1)$$

F-distribution

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k {
- $\{Y_{1;1}, Y_{1;2}, \ldots\ldots, Y_{1;N1}\}$ IID from a Normal($\mu_1;\sigma_1$)
- $\{Y_{2;1}, Y_{2;2}, \ldots, Y_{2;N2}\}$ IID from a Normal($\mu_2;\sigma_2$)
- ,…
- $\{Y_{k;1}, Y_{k;2}, \ldots, Y_{k;N2}\}$ IID from a Normal($\mu_2;\sigma_2$)
}
- $\sigma_1 = \sigma_2 = \sigma_3 = \ldots \sigma_{n_k} = \sigma$. ($1/2 \le \sigma_k/\sigma_j \le 2$)
- Samples are independent!

F-distribution

- F-distribution k-samples of different sizes

Typical Analysis-of-Variance Table for One-Way ANOVA

Source	Sum of squares	*df*	Mean sum of Squares[a]	*F*-statistic	*P*-value
Between	$\sum n_i(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2$	k - 1	s_B^2	$f_0 = s_B^2 / s_W^2$	pr($F \ge f_0$)
Within	$\sum (n_i - 1)s_i^2$	n_{tot} - k	s_W^2		
Total	$\sum\sum (x_{ij} - \bar{x}_{\cdot\cdot})^2$	n_{tot} - 1			

[a]Mean sum of squares = (sum of squares)/*df*

$$s_B^2 = \frac{\sum n_i(\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2}{k-1}$$

$$s_W^2 = \frac{\sum (n_i - 1)s_i^2}{n_{tot} - k}$$

- s^2_B is a measure of variability of sample means, how far apart they are.
- s^2_W reflects the avg. internal variability within the samples.

Development of Test Statistic

MSTr = Mean Sum-square due to Treatment describes "**between-samples**" variation

$$MSTr = \frac{J}{I-1}\sum_{i=1}^{I}\left(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot}\right)$$

MSE = Mean Sum-square due to Error describes "**within-samples**" variation

$$MSE = \frac{S_1^2 + S_2^2 + \ldots + S_I^2}{I}$$

Computational Formulas Cont'd

Identity: SST = SSTr + SSE

- Partition total variation into two pieces
- SSE (within) measures variation that would be present even if H_o true (unexplained by H_o when true or false)
- SSTr (between) measures amount of variation that can be explained by possible differences in the μ_i's (explained by H_o when false)

Example

One manufacturing firm in interested in the concentration of impurities in steel obtained from 4 different vendors. Test the hypothesis that the mean concentration of impurities is the same for all vendors at a 0.01 level of significance (LOS).

Example Data: I=4, J=10

Demo: SYSTAT →CopyNPasteData_Sheet2 →Statistics→ANOVA

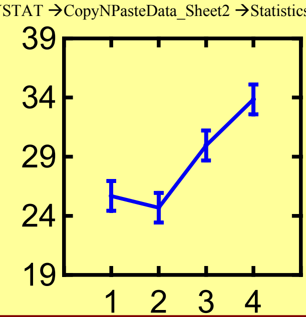
Vendor1	Vendor 2	Vendor 3	Vendor 4
20.5	26.3	29.5	36.5
28.1	24	34	44.2
27.8	26.2	27.5	34.1
27	20.2	29.4	30.3
28	23.7	27.9	31.4
25.2	34	26.2	33.1
25.3	17.1	29.9	34.1
20.5	26.8	29.5	32.9
31.3	23.7	30	36.3
23.1	24.9	35.6	25.5

Example Data: I=4, J=10

- Demo: SYSTAT →CopyNPasteData_Sheet2 →Statistics→ANOVA
- `Categorical values encountered are:`
- `INDEX (4 levels)1,   2,    3,   4`
- `Dep Var: VAR00002    N: 40    Multiple R: 0.69460   Squared multiple R: 0.48247`
- **`Analysis of Variance`**
- `Source   SS  df Mean-Square F-ratio    P`
- `INDEX 530.80200 3 176.9  11.18   0.00002`
- `Error 569.37400 36 15.8`

Example Data: I=4, J=10

- Demo: SYSTAT →CopyNPasteData_Sheet2 →Statistics→ANOVA



Multiple Comparisons (10.2)

Assume that the null hypothesis of a single-factor ANOVA test is rejected.

H_o: $\mu_1 = \mu_2 = \ldots = \mu_n$

H_a: at least two of the μ_i's differ

Which μ_i's differ?

Use one of: Least Significant Difference Procedure, Tukey's Procedure, Newman-Keuls Procedure, Duncan's Multiple Range Procedure

Tukey's Procedure (Conservative) –T Method

- Used to obtain simultaneous confidence intervals for all pair-wise differences $\mu_i - \mu_j$
- Each interval that does not contain zero yields the conclusion that μ_i and μ_j differ significantly at level α
- Based on the Studentized Range Distribution, $Q_{\alpha,m,\nu}$; m=d.f. numerator, ν = d.f. of deno; for Tukey's Proc. m = I, ν=I(J-1)

Tukey's Procedure Cont'd

1. Select α and find $Q_{\alpha,I,I(J-1)}$, using tables or SOCR
2. Determine $w = Q_{\alpha,I,I(J-1)} (MSE/J)^{1/2}$
3. List the sample means in increasing order. Underline those **pairs that differ by less than w**. Any pair not underscored by the same line are judged **significantly different.**

Example (10.11)

Compare the spreading rates of (I=5) different brands of Latex paint using (J=4) gallons of each paint. The sample average spreading rates were

$$\bar{x}_{1\cdot} = 462.0,\ \bar{x}_{2\cdot} = 512.8$$

$$\bar{x}_{3\cdot} = 437.5,\ \bar{x}_{4\cdot} = 469.3,$$

$$\bar{x}_{5\cdot} = 532.1,\ \bar{x}_{\cdot\cdot} = 482.8$$

Example Cont'd

From an ANOVA test on the equality of means, the computed value of F was found to be significant at $\alpha = 0.05$ with MSE = 272.8, use Tukey's procedure to investigate significant differences in the true average spreading rates between brands.

MSTr= 5,900/4 = 1475

F=MSTr/MSE = 5.4 $\sim F_{(0.05, 4, 20\text{-}5)}$ ➔
SOCR P-value = 0.006746436876727799 ➔ signif.

Example Cont'd

MSTr= 5,900/4 = 1475

F=MSTr/MSE = 5.4 $\sim F_{(0.05, 4, 20\text{-}5)}$ ➔
SOCR P-value = 0.006746436876727799 ➔ signif.

Five sample means in increasing order:

$\bar{x}_{3\cdot} = 437.5$, $\bar{x}_{1\cdot} = 462.0$, $\bar{x}_{4\cdot} = 469.3$, $\bar{x}_{2\cdot} = 512.8$, $\bar{x}_{5\cdot} = 532.1$

------------------------------------- ↑ ----------------------

w= $Q_{0.05, 5, 15}\,(272.8/4)^{1/2} = 4.37 \times 8.3 = 36.1$

A Caution About Interpreting α

α` = experiment wise error rate. This is the confidence level for the entire set of comparisons of means

α = comparison wise error rate. This is the confidence level for any particular individual comparison.

α` = Pr{at least 1 false rejection among the c comparisons} = 1 – Pr{no false rejections} = $1-(1-\alpha)^c$

Example Cont'd

We used Tukey's procedure to compare 5 different population (α=0.05) means resulting in

$\binom{5}{2} = 10 = c$ pairwise comparisons of means

$$\alpha` = 1-(1-.05)^{10} = .59$$

Real error if no correction (Tukey) is applied!

Contrasts

- Elementary Contrasts: $\mu_1 - \mu_2$
- General Contrasts: $c_1\mu_1 + c_2\mu_2 + \ldots + c_n\mu_n$; where $c_1+c_2+\ldots+c_n=0$

We would like to form a CI on a general contrast, For example, construct a CI on the contrast $\mu_1 + \mu_2 - 2\mu_3$

Contrasts (cont'd)

Let $\theta = \Sigma c_i \mu_I$. Since the X_{ij}'s are (independent) normally distributed and the contrast is a linear combination, $\hat{\theta} = \sum c_i \overline{X}_{i\cdot}$ is normally distributed since

$$f_{\hat{\theta}}(\theta) \underset{IID}{=} \prod_{k=1}^{n} f_{X_K}(x) = \prod_{i=1}^{n} \exp\left(\frac{(x_K-\mu)^2}{2\sigma^2}\right) =$$

$$\exp\left(\frac{\sum_{k=1}^{n}(x_K-\mu)^2}{2\sigma^2}\right) = \exp\left(\frac{\|x-\mu\|_2^2}{2\sigma^2}\right)$$

Example (cont'd)

Assume that brands 2 and 5 were bought at a local paint store and 1, 3, and 4 were bought at a discount hardware store. Is there evidence that the quality of paint varies by type (classification) of store?

Interpreting α and α` for Multiple Comparisons Revisited

α` = "experiment wise error rate" =

= "composite error rate"

α` = Pr{at least 1 false rejection among the ***c*** comparisons} =

= 1 – Pr{no false rejections} = $1-(1-\alpha)^c$

• In obtaining the above expression, we assumed that each of the ***c*** comparisons was independent

Interpreting α and α` for Multiple Comparisons Revisited

• These ***c*** comparisons, however, generally are dependent

• It follows that the α` computed previously assuming independence serves as an upper bound to the "True" experiment wise error rate that accounts for the dependence between the ***c*** comparisons.

Single-Factor ANOVA – Sample Sizes Unequal

• Let $J_1, J_2, \ldots, J_n$ denote the I sample sizes

• Let the total number of observations $n = \Sigma_i J_i$

Example (10.26)

Samples of six different brands of imitation **margarine** were analyzed to determine the level of PAP fatty acids (pyelonephritis-associated pilus).

Use ANOVA to test for differences among the true average PAP fatty acids percentages for the different brands

Example (10.26)

Imperial®, 14.1, 13.6, 14.4, 14.3

Parkay®, 12.8, 12.5, 13.4, 13.0, 12.3

Blue Bonnet®, 13.5, 13.4, 14.2, 14.3

Chiffon®, 13.2, 12.7, 12.6, 13.9

Mazola®, 16.8, 17.2, 16.4, 17.3, 18.0

Fleischmann's®, 18.1, 17.2, 18.7, 18.4

Mazola and Fleischmann's are corn-based where the others are soybean-based.

Multiple Comparisons when Sample Sizes are Unequal

- Use the following modified Tukey's procedure when the I sample sizes $J_1, J_2, \ldots, J_I$ are reasonably close.
- The computed w_{ij} depends on J_i and J_j respectively. That is, each $CI(\mu_i - \mu_j)$ has an associated w_{ij} that varies between i and j according to their respective sample size.

Example Cont'd

- Use the modified Tukey's procedure to determine which means differ
- $w_{i,j} = Q_{\alpha, I, n-I} (MSE \times (1/J_i + 1/J_j) / 2)^{1/2}$
- Then

$$1-\alpha = \Pr\left(\overline{X_{i.}} - \overline{X_{j.}} - w_{i,j} \le \mu_i - \mu_j \le \overline{X_{i.}} - \overline{X_{j.}} + w_{i,j}\right)$$

Example Cont'd

Compute a C.I. for the contrast

$$\theta = \frac{(\mu_1+\mu_2+\mu_3+\mu_4)}{4} - \frac{(\mu_5+\mu_6)}{2} \Rightarrow$$

$$\hat{\theta} = \frac{(\bar{x}_1+\bar{x}_2+\bar{x}_3+\bar{x}_4)}{4} - \frac{(\bar{x}_5+\bar{x}_6)}{2} =$$

$$\hat{\theta} = \sum_{k=1}^{n} c_k x_k \Rightarrow$$

$$CI(\theta) = \hat{\theta} \pm \sqrt{(I-1)\times MSE \times F_{a,I=1,n-I}} \times \sqrt{\sum_{k=1}^{n} c_k^2 / J_k}$$

Model Equation

$$X_{ij} = \mu_i + \varepsilon_{ij}$$

- Alternative description of ANOVA model

X_{ij} = each observation or response

μ_i = the mean of the i^{th} population or treatment

ε_{ij} = deviation of the j^{th} observation from the i^{th} population or treatment mean

Model Equation

- Assume that ε_{ij} are independent and normally distributed RV's such that $E[\varepsilon_{ij}] = 0$ and $Var[\varepsilon_{ij}] = \sigma^2$, i.e., $\varepsilon_{ij} \sim N(0, \sigma^2)$.
- It follows that: $X_{ij} \sim N(\mu_i, \sigma^2)$ as specified by the ANOVA assumptions.

Linear Model

Define a new parameter μ by:

$$\mu = \frac{1}{I}\sum_{i=1}^{I} \mu_i$$

Define new parameters $\alpha_1, \ldots, \alpha_n$ by:

$$\alpha_i = \mu_i - \mu$$

Linear Model

• Expressing the model equation in terms of these new parameters yields

$$X_{ij} = \mu + \alpha_i + \varepsilon_{ij} \; ; \; \Sigma \alpha_i = 0$$

• The null hypothesis for the ANOVA test that H_o: $\mu_1 = \ldots = \mu_I$ is equivalent to H_o: $\alpha_1 = \ldots = \alpha_I$

Fixed vs. Random Effects

• **Fixed Effects Model** – The experiment was conducted using all treatments of interest to the researcher

• **Random Effects Model** – A researcher wants to inferences about a set of treatments larger than that used in the sample. The treatments used in the experiment represent a random sample of all treatments of interest

Fixed vs. Random Cont'd

• Fixed effects model: α_i's are unknown parameters

• Random effects model: Replace α_i's with A_i's where $E[A_i]=0$ and $Var[A_i]=\sigma^2$.

• The ANOVA test for Fixed and Random effects models does not differ, even though the form of the null hypothesis does.

ANOVA Assumptions

Consider the linear model $X_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

i) μ is a fixed constant common to all observations

ii) The ε_{ij} are independent and normally distributed with $E[\varepsilon_{ij}]=0$ and $Var[\varepsilon_{ij}]=\sigma^2$

iii) The deviations from the overall mean for the I treatments are such that $\Sigma \alpha_i = 0$

ANOVA Assumptions

Under these assumptions:

• $E[X_{ij}]=\mu_i$

• $Var[X_{ij}]=\sigma^2$

• and X_{ij} is normally distributed

which facilitates the use of ANOVA for testing hypothesis about the equality of the means

ANOVA Assumptions

In real world experiments, however, either the normality and/or equal variances assumptions are often violated. How robust is the ANOVA test to these violations?

Normality Assumption

- It was established by Cochran and Hay that the ANOVA test is very robust with respect to non-normality.
- Regardless, the plausibility of a normal assumption for X_{ij} under a fixed i may be established through Normal Probability Plots (NPP) or quantile-quantile plots (Q-Q Plot)

Normal Probability Plots (4.6)

A NPP is a plot of the observed data values against the z-percentiles of the standard normal distribution.

- If the plotted points do not deviate greatly from a straight 45^o line, then it is plausible to assume that our data is normally distributed

Normal Probability Plots (4.6)

- If the plotted points fall in an S shape, then it is plausible to assume that our data from a **heavy-tailed distribution**
- If the plotted points fall in a backwards S shape, then it is plausible to assume that our data from a light-tailed distribution
- If the plotted points fall in a middle curved shape, then it is plausible to assume that our data from a positively skewed distribution

Equal Variances Assumption

- It was established by Welsh and Box that the ANOVA procedure is robust to mild departures from the equal variances assumption for equal replications
- If there is a large departure from the equal variances assumption and/or mild departures with extremely unequal replications, a variance stabilizing transformation should be used if possible

Variance Stabilizing Data Transformations

If Var $[X_{ij}] = g(\mu_i)$ (that is the variance is a known function of the mean) then the transformation $h(X_{ij})$ such that Var $[X_{ij}]$ is approximately the same for each ***i*** is given by

$$h(x) \propto \int [g(x)]^{1/2} dx$$

Common Transformations

Common variance stabilizing transformations

If the response is a Poisson count, so that the variance is proportional to the mean, use the **square root transformation**:

$$y' = y^{1/2} = \sqrt{y}$$

If the response is a binomial proportion, use the **arcsine square root transformation**:

$$\hat{p}' = \sin^{-1}\left(\sqrt{\hat{p}}\right)$$

Common variance stabilizing transformations

If the variance is proportional to the mean squared, use the **natural log transformation**:

$$y' = \log_e(y)$$

If the variance is proportional to the mean to the fourth power, use the **reciprocal transformation**:

$$y' = -\frac{1}{y}$$

Knowing functional relationship is of the power form

If the relationship between x and y is of the **power form**:

$$y = \alpha x^{\beta}$$

taking log of both sides transforms it into a linear form:

$$\log_e y = \log_e \alpha + \beta \log_e x$$

Knowing functional relationship is of the exponential form

If the relationship between x and y is of **exponential form**:

$$y = \alpha e^{\beta x}$$

taking log of both sides transforms it into a linear form:

$$\log_e y = \log_e \alpha + \beta x$$

Further Comments on Data Transformations

Does a data transformation destroy the other needed properties such as normality and independence?

Answer: Generally No! In fact, the presence of non-normality and unequal variances are often related. It has been shown that transformations to stabilize the variance often helps to correct non-normality in the data

Example

A small restaurant chain has 4 different locations in the local area. The management is interested in whether the true average of complaints received per restaurant differs by location. The number of complaints at each restaurant was counted and recorded for 30 consecutive months. Test the appropriate hypothesis at α=0.05 los.

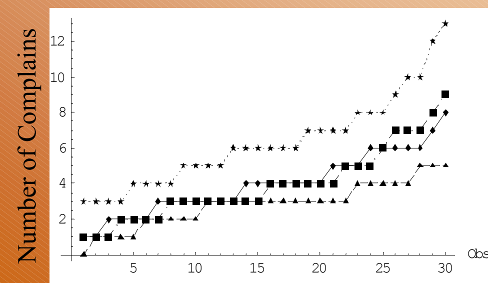
Location1:{1,1,2,2,2,2,3,3,3,3,3,3,3,4,4,4,4,4,4,4,5,5,5,6,6,6,6,6,7,8}

Location2:{3,3,3,3,4,4,4,4,5,5,5,5,6,6,6,6,6,7,7,7,7,8,8,8,9,10,10,12,13}

Location3:{1,1,1,2,2,2,2,3,3,3,3,3,3,3,3,4,4,4,4,4,4,5,5,5,6,7,7,7,8,9}

Location4:{0,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3,3,3,3,3,3,3,4,4,4,4,4,5,5,5}

Plot of the Data Sets



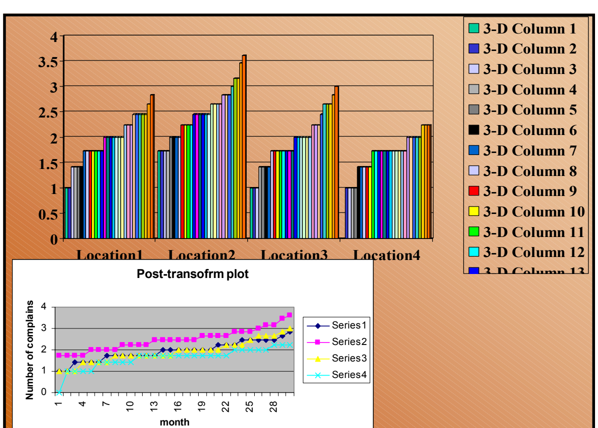
Transformed ($x^{0.5}$) Data Sets

Location1:{1,1,1.41421,1.41421,1.41421,1.41421,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,2.,2.,2.,2.,2.,2.,2.,2.23607,2.23607,2.23607,2.44949,2.44949,2.44949,2.44949,2.44949,2.64575,2.82843}

Location2:{1.73205,1.73205,1.73205,1.73205,2.,2.,2.,2.,2.23607,2.23607,2.23607,2.23607,2.44949,2.44949,2.44949,2.44949,2.44949,2.44949,2.64575,2.64575,2.64575,2.64575,2.82843,2.82843,2.82843,3.,3.16228,3.16228,3.4641,3.60555}

Location3:{1,1,1,1.41421,1.41421,1.41421,1.41421,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,2.,2.,2.,2.,2.,2.,2.23607,2.23607,2.23607,2.44949,2.64575,2.64575,2.64575,2.82843,3.}

Location4:{0,1,1,1,1,1.41421,1.41421,1.41421,1.41421,1.41421,1.41421,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,1.73205,2.,2.,2.,2.,2.,2.23607,2.23607,2.23607}



Two- Factor ANOVA K_{ij}=1 (11.1)

- Two Factors of Interest (A) and (B)
- I = number of levels of factor A
- J = number of levels of factor B
- K_{ij} = number of observations made on treatment (i,j)

Example

Consider an experiment to test the effect of heat and pressure on the strength of a steel specimen. Specifically, the test will consider the temperatures 100,120,130,140 degrees Celsius and the pressures 100,150,200 psi. Each temp/pressure combination will be observed once

- Factor A = Temp, B =Pressure
- $I=4$, $J=3$, $K_{ij}=1$

The Model

$$X_{ij}=\mu_{ij} + \varepsilon_{ij}$$

- This model has more parameters than observations
- A unique additive (no interactions) linear model is given by

$$X_{ij}=\mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

Where $\Sigma\alpha_i=0$, $\Sigma\beta_j=0$, $\varepsilon_{ij}\sim N(0,\sigma^2)$

Additive Model

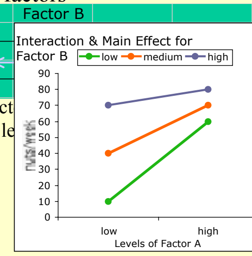
- Necessary assumption since $K_{ij}=1$
- The difference in mean responses for two levels of factor A(B) is the same for all levels of factor B(A); i.e. The difference in the mean responses for two levels of a particular factor is the same regardless of the level of the other factor

Plots for Checking Additivity (no interactions)

- When the effect of one factor depends on the different levels of a second factor, then there is an **interaction** between the factors

Factor A		Factor B
	low	
low	10	
high	60	
	35	

- Similarly, the effect of fact
  but is only 20 at the high le



- **If the lines are NOT parallel, there IS an interaction**

Interpretation of the Model

- μ = The true grand mean
- α_i = The effect of factor A at level i
- β_j = The effect of factor B at level j

$$\hat{\mu} = \overline{X}_{\cdot\cdot}$$
$$\hat{\alpha}_i = \overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot}$$
$$\hat{\beta}_j = \overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot}$$

Hypothesis of Interest

1. H_{oA}: $\alpha_1 = \alpha_2 = \ldots = \alpha_i = 0$

   H_{aA}: at least one $\alpha_i \neq 0$

2. H_{oB}: $\beta_1 = \beta_2 = \ldots = \beta_j = 0$

   H_{aB}: at least one $\beta_j \neq 0$

Multiple Comparisons

Use only after H_{oA} and/or H_{oB} has been rejected

Example – Two-Factor ANOVA (11.2)

A study on the type of coating and type of soil on the corrosion of a metal pipe is considered (4 types of coatings (A) and 3 types of soil (B)). 12 pieces of pipes are selected and each receives one of the factor level combinations. After a fixed time, the amount of corrosion is measured for each pipe. The data is as follows:

Randomized Block Experiments

Under single-factor ANOVA, we assumed that our IJ experimental units are homogeneous with respect to other variables that may affect the observed response

If there is heterogeneity, however, the calculated F may be affected by these other variables; use blocking to “block out” this extraneous variation

Blocking Cont’d

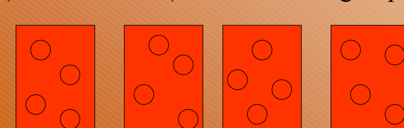
- Form “blocks” such that the units are homogeneous within each group (block) with respect to the extraneous factor
- Divide the IJ units into J groups (blocks) with I units in each group.
- Within each homogenous group (block), the I treatments are randomly assigned to the I units
- When I=2, either the paired t-test or F test may be used, the results are the same

Example “Blocking”

A soil and crops scientist is interested in comparing the effect of four different types of fertilizer on the yield of a specific type of corn. He has 4 different plots of land (each sub dividable into 4 lots) at his disposal scattered throughout the state. The ph level of the soil is know to affect the yield of corn and this varies at each plot.

Example Cont’d

- I=4 (types of fertilizer – A,B,C,D)
- Block on soil PH level, I.e J=4 groups with the I=4 treatments assigned to I=4 units (subdivided lots) at within each group



Example Cont'd

$$X_{ij}=\mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

- α_i = effect of the fertilizer factor at level i (deviations due to fertilizer factor at level i)
- β_j = effect of the block at level j (variability by block)
- ε_{ij} = random error of the i,jth observation (variability around the block)

Additional Comments on Blocking

Blocking may reduce the value of the parameter σ^2 as estimated by the MSE, resulting in a larger calculated f test statistic

The probability of a type II error is decreased, however, only if the gain in the calculated f offsets the loss in the denominator degrees of freedom for the critical F value; that is I(J-1) d.f under single-factor ANOVA vs. (I-1)(J-1) under blocked two-factor ANOVA

Additional Comments on Blocking Cont'd

- If the number of IJ observations is small, care should be taken in deciding whether blocking is warranted in reducing the Type II error probability

Example – Blocking (11.6)

A particular county has 3 assessors who determine the value of residential property. To test whether the assessors systematically differ, 5 houses are selected and each assessor is asked to determine their value. Explain why blocking is used in this experiment rather than a one-way ANOVA test

Random Effects Model

Fixed Effects Model:

$$X_{ij}=\mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

Random Effects Model:

$$X_{ij}=\mu + A_i + B_j + \varepsilon_{ij}$$

- $A_i \sim N(0,\sigma_A^2)$
- $B_j \sim N(0,\sigma_B^2)$
- $\varepsilon_{ij} \sim N(0,\sigma^2)$

Random Effects Model Cont'd

Hypotheses:

$$H_{oA}: \sigma_A^2 = 0,\ H_{oB}: \sigma_B^2 = 0$$

$$H_{aA}: \sigma_A^2 > 0,\ H_{aB}: \sigma_B^2 > 0$$

- $E(MSA) = \sigma^2 + J\,\sigma_A^2$
- $E(MSB) = \sigma^2 + J\,\sigma_B^2$ $\quad f_A = E(MSA) / E(MSE)$
- $E(MSE) = \sigma^2$ $\quad f_B = E(MSB) / E(MSE)$

Mixed Effects Model

$X_{ij} = \mu + \alpha_i + B_j + \varepsilon_{ij}$

$H_{oA}: \alpha_1 = \ldots = \alpha_n = 0$, $H_{oB}: \sigma_B^2 = 0$

H_{aA}: at least one α_i differs, $H_{aB}: \sigma_B^2 > 0$

- $E(MSA) = \sigma^2 + (J/I-1)\Sigma\alpha_i$
- $E(MSB) = \sigma^2 + J\sigma_B^2$
- $E(MSE) = \sigma^2$

$f_A = E(MSA) / E(MSE)$

$f_B = E(MSB) / E(MSE)$

Example – Blocking (Fixed and Random Effects) (11.6,12)

A particular county has 3 assessors who determine the value of residential property. To test whether the assessors systematically differ, 5 houses are selected and each assessor is asked to determine their value. Let factor A denote the assessor and factor B denote the the houses. We compute SSA=11.7, SSB=113.5, and SSE = 25.6

Example Cont'd

Suppose that the 6 houses in the previous example had been selected at random from among those of a certain age and size. It follows that factor B is random rather than fixed

Two-Factor ANOVA, $K_{ij}>1$ (11.2)

- When $K_{ij}>1$, an estimator of the the variance σ^2 (MSE) of ε may be obtained without assuming additivity.
- This allows for our model to include an interaction parameter
- Assume that $K_{ij} = K > 1$ for all i,j

The Model

Let :

- μ_{ij} = The true average response when factor A is at level i and factor B at level j
- $\mu = (\Sigma_j \Sigma_j \mu_{ij})/IJ$ = The true grand mean
- $\mu_{i.} = (\Sigma_j \mu_{ij})/J$ = The expected response of factor A at level i averaged over factor B
- $\mu_{.j} = (\Sigma_i \mu_{ij})/I$ = The expected response of factor B at level j averaged over factor A

The Model Cont'd

- $\alpha_i = \mu_{i.} - \mu$ = The effect of factor A at level i (main effects for factor A)
- $\beta_j = \mu_{.j} - \mu$ = The effect of factor B at level j (main effects for factor B)
- $\gamma_{ij} = \mu_{ij} - (\mu + \alpha_i + \beta_j)$ = interaction effect of factor A at level i and factor B at level j (interaction parameters)

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

The Model Cont'd

$X_{ijk} = \mu + \alpha_i + B_j + \gamma_{ij} + \varepsilon_{ijk}$, $\varepsilon_{ijk} \sim N(0,\sigma^2)$

Hypotheses:

$H_{oAB}: \gamma_{ij} = 0$, H_{aAB} = at least one $\gamma_{ij} \neq 0$

$H_{aA}: \alpha_1 = \ldots = \alpha_n = 0$, H_{aA}: at least one $\alpha_i \neq 0$

$H_{aB}: \beta_1 = \ldots = \beta_n = 0$, H_{aB}: at least one $\beta_i \neq 0$

The Test

- Test the no-interaction hypothesis H_{oAB} first
- If H_{oAB} is not rejected
  - Test the other hypothesis H_{oA} and H_{oB}
- If H_{oAB} is rejected
  - Do not test the other hypothesis H_{oA} and H_{oB}
  - Construct an interaction plot to visualize how the factors interact

The Test Cont'd

Assume that we reject H_{oAB} and then go on to test H_{oA} and H_{oB}. Suppose that H_{oA} is rejected. The resulting model would be

$$\mu_{ij} = \mu + \alpha_j + \gamma_{ij}$$

which does not have a clear interpretation. In other words, an insignificant main effect has little meaning in the presence of a significant interaction effect.

The Test Cont'd

- $E(MSA) = \sigma^2 + (JK/I-1)\, \Sigma\alpha_i^2$
- $E(MSB) = \sigma^2 + (IK/J-1)\, \Sigma\beta_i^2$
- $E(MSAB) = \sigma^2 + [K/((I-1)(J-1))]\, \Sigma\Sigma\gamma_{ij}^2$
- $E(MSE) = \sigma^2$

$f_A = E(MSA) / E(MSE)$

$f_B = E(MSB) / E(MSE)$

$f_{AB} = E(MSAB) / E(MSE)$

ANOVA Table

Example (11.19)

The accompanying data gives observations of the total acidity of coal samples of three different types, with determinations made using three different concentrations of **sodium hydroxide** NaOH. Assuming fixed effects, construct an ANOVA table and test for the presence of interactions and main effects at los 0.01

Example (11.19)

The accompanying data gives observations of the total acidity of coal samples of three different types, with determinations made using three different concentrations of NaOH. Assuming fixed effects, construct an ANOVA table and test for the presence of interactions and main effects at LOS 0.01

Multiple Comparisons

* Use if H_{oAB} is not rejected and either or both of H_{oA} and H_{oB} are rejected *

To test for differences of the α_i's when H_{oA} is rejected

1. Obtain $Q_{\alpha,I,IJ(K-1)}$
2. Compute $w = Q(MSE/(JK))^{1/2}$
3. Order the $\bar{x}_{i\cdot\cdot}$ from the smallest to largest and proceed with the underlining method

To test for differences of the β_j's when H_{oB} is rejected

1. Obtain $Q_{\alpha,J,IJ(K-1)}$
2. Compute $w = Q(MSE/(IK))^{1/2}$
3. Order the $\bar{x}_{\cdot j\cdot}$ from the smallest to largest and proceed with the underlining method

Example Cont'd (11.19)

Use Tukey's procedure to identify significant differences among the types of coal

Mixed Effects Model

The methods developed under mixed effects will naturally extend to the random effects model

$$X_{ijk} = \alpha_i + B_j + G_{ij} + \varepsilon_{ijk}$$

- α_i = Fixed effect of Factor A at level I, $\Sigma \alpha_i = 0$
- B_j = Random effect of Factor B at level j, $B_j \sim N(0,\sigma_B^2)$

- G_{ij} = Interaction effect of Factor A at level i and Factor B at level j , $G_{ij} \sim N(0, \sigma_G^2)$
- ε_{ijk} = Random error of the k^{th} observation with Factor A at level i and Factor B at level j

Hypotheses of Interest

- $H_{oA}: \alpha_1 = \ldots = \alpha_I = 0$; H_{aA}: at least one $\alpha_i \neq 0$
- $H_{oB}: \sigma_B^2 = 0$; $H_{aB}: \sigma_B^2 > 0$
- $H_{oG}: \sigma_G^2 = 0$; $H_{aB}: \sigma_G^2 > 0$

* Test H_{oA} and H_{oB} only if H_{oG} is not rejected*

Development of Test

Compute the Sums of Squares, Mean Squares, and ANOVA table identically to that under fixed effects

- $E[MSE] = \sigma^2$
- $E[MSA] = \sigma^2 + K\,\sigma_G^2 + (JK/I\text{-}1)\,\Sigma\,\alpha_i^2$
- $E[MSB] = \sigma^2 + K\,\sigma_G^2 + IK\,\sigma_B^2$
- $E[MSAB] = \sigma^2 + K\,\sigma_G^2$

Test of H_{oG}

$F_{ab} = E[MSAB]/E[MSE] = (\sigma^2 + K\,\sigma_G^2)/\,\sigma^2$

- Under H_{oG}: $f_{ab} = 1$
- Under H_{aG}: $f_{ab} = 1 + (K\,\sigma_G^2/\sigma^2) > 1$ for $\sigma_G^2 > 0$

Reject H_{oG} if $f_{ab} > F_{\alpha,(I-1)(J-1),IJ(K-1)}$

If we fail to reject H_{oG} then test H_{oA} and H_{oB}

Test of H_{oA}

$F_A = E[MSA]/E[MSAB]$

$= (\sigma^2 + K\,\sigma_G^2 + (JK/I\text{-}1)\,\Sigma\,\alpha_i^2)/(\sigma^2 + K\,\sigma_G^2)$

Notice that the denominator of F_A is E[MSAB]; not E[MSE]

- Under H_{oA}: $f_A = 1$
- Under H_{aA}: $f_A = 1 + [(JK/I\text{-}1)\,\Sigma\,\alpha_i^2)/(\sigma^2 + K\,\sigma_G^2)] > 1$ for $\Sigma\,\alpha_i \neq 0$

Reject H_{oA} if $f_A > F_{\alpha,I-1,(I-1)(J-1)}$

Test of H_{oB}

$F_B = E[MSB]/E[MSAB]$

$= (\sigma^2 + K\,\sigma_G^2 + IK\,\sigma_B^2)/(\sigma^2 + K\,\sigma_G^2)$

Again the denominator of F_B is E[MSAB]; not E[MSE]

- Under H_{oA}: $f_B = 1$
- Under H_{aA}: $f_B = 1 + [(IK\,\sigma_B^2)/(\sigma^2 + K\sigma_G^2)] > 1$ for $\Sigma\,\alpha_i \neq 0$

Reject H_{oB} if $f_B > F_{\alpha,J-1,(I-1)(J-1)}$

Example (11.19 modified)

Assume that the determinations for the level of acidity of the three different types of coal were to made using 3 levels of a **sodium** hydroxide NaOH factor that could range between 0N and 1N. We randomly choose the concentrations .404N, .626N, and .786N

Three Factor ANOVA

I,J,K = Levels of the factors A,B, C

L_{ijk} = The number of obeservations of factor A at level i, factor B at level j, and factor C at level k

L_{ijk} = L for all i,j,k – Equal replications for all factor level combinations

The Model

$$X_{ijkl} = \mu_{ijk} + \varepsilon_{ijkl}$$

I = 1,…,i; J=1,…,j;
K=1,…,k; L=1,…,l

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC} + \gamma_{ijk}$$

$\gamma_{ij}^{AB}, \gamma_{ik}^{AC}, \gamma_{jk}^{BC}$ = Two Factor Interactions

γ_{ijk} = Three Factor Interactions

$\alpha_i + \beta_j + \delta_k$ = Main Effects

Interpretation of Interactions

The interaction between factor A at level i and factor B at level j for factor C at level k

$$\mu_{ijk} - \mu_{i\cdot k} - \mu_{\cdot jk} + \mu_{\cdot\cdot k} = \gamma_{ij}^{AB} + \gamma_{ijk}$$

The interaction between factor A at level i and factor B at level j averaged over all levels of factor C

$$\mu_{i\cdot k} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu_{\cdot\cdot\cdot} = \gamma_{ij}^{AB}$$

ANOVA Table

* calculate the Sums of Squares using a computer *

Source	df	Sums of Squares	Mean Square	f
A	I-1	SSA	MSA	MSA/MSE
B	J-1	SSB	MSB	MSB/MSE
C	K-1	SSC	MSC	MSC/MSE
AB	(I-1)(J-1)	SSAB	MSAB	MSAB/MSE
AC	(I-1)(K-1)	SSAC	MSAC	MSAC/MSE
BC	(J-1)(K-1)	SSBC	MSBC	MSBC/MSE
ABC	(I-1)(J-1)(K-1)	SSABC	MSAC	MSAC/MSE
Error	IJK(L-1)	SSE	MSE	
Total	IJKL-1	SST		

Test

1. First, test for the presence of three factor interactions
2. If these are deemed not significant, test for the presence of two factor interactions

- If these are judged not significant, test for the presence of the main effects
- If some or all of these are deemed significant, construct interaction plots. (If all two factor interaction effects are significant, the plots may be difficult to interpret)

Multiple Comparisons

Use Tukey's Procedure to perform a pair wise comparisons of the means of a significant factor

1. Find Q with the first d.f. equal to the number of means being compared and the second d.f equal d.f. for the error = IJK(L-1)
2. Compute $w = Q(MSE/N)^{1/2}$ where N = JKL for comparing factor A, N=IKL for comparing factor B, N=JKL for comparing factor C
3. Order the means and perform the underlining procedure

Example

Latin Squares

• Complete Layout – At least one observation for each factor level combination

• Incomplete Layout – Fewer than one observation for each factor level combination

- A Latin Square is a type of incomplete layout that may be analyzed in a straightforward fashion

Significance of Latin Squares

• Focuses on the main effects

•A complete layout for a three factor ANOVA with one observation at each of the IJK=N factor-level combinations would require N^3 observations. A Latin Square layout would require only N^2 observations. If I=J=K=4, the complete layout would require 64 observations, the Latin Square would require 16 observations. If data collection is costly, this may significantly reduce time and costs.

Assumptions of Latin Squares

• Each factor has the same number of levels I=J=K with no more than one observation at any particular factor-level combination

•The model is completely additive – No significant two or three factor interaction effects (This is a strong assumption)

• Both the square used and observations in the square are taken at random

Construction of Latin Squares

Consider a table where

- Rows = Levels of Factor A
- Columns = Levels of Factor B

A Latin Square prescribes that every level of factor C appears exactly once in each row and column.

Construction of Latin Squares Cont'd

There are 12 different 3x3 Latin Squares, the number of squares increases rapidly with N

Example

Suppose a chemical company is interested in testing the burning rate of 3 different formulations of rocket propellant. There are 3 different batches of raw materials from which each formulation is mixed, and there are 3 different lab technicians that prepare the batches whose experience greatly differs.

The Model

$X_{ij(k)} = \mu + \alpha_i + \beta_j + \delta_k + \varepsilon_{ij(k)}$ i,j,k=1,…,n

ANOVA Table

Multiple Comparisons

Use Tukey's Procedure

1. Find $Q_{\alpha,N,(N-1)(N-2)}$
2. Compute $w = Q(MSE/N)^{1/2}$
3. Order the means and perform the underlining procedure

Example

Suppose a chemical company is interested in testing the burning rate of 3 different formulations of rocket propellant. There are 3 different batches of raw materials and 3 lab technicians, whose experience greatly differs, that prepare the formulations.

The Model

$X_{ij(k)} = \mu + \alpha_i + \beta_j + \delta_k + \varepsilon_{ij(k)}$ i,j,k=1,…,n

ANOVA Table

Source	d.f.	SS	MS	f
A (rows)	N-1	SSA	MSA	MSA/MSE
B (columns)	N-1	SSB	MSB	MSB/MSE
C (trts)	N-1	SSC	MSC	MSC/MSE
Error	(N-1)(N-2)	SSE	MSE	
Total	N^2-1	SST		

Multiple Comparisons

Use Tukey's Procedure

1. Find $Q_{\alpha,N,(N-1)(N-2)}$
2. Compute $w = Q(MSE/N)^{1/2}$
3. Order the means and perform the underlining procedure

Example (11.34)

Consider an experiment in which the effect of shelf space on food sales is investigated. The experiment was conducted over a 6 week period using 6 different stores. Assuming no interactions, construct a Latin Square for this experiment

The data collected for this experiment follows. Test the hypothesis that shelf space does not affect sales at a .01 los.

	1	2	3	4	5	6
1	27(5)	14(4)	18(3)	35(1)	28(6)	22(2)
2	34(6)	31(5)	34(4)	46(3)	37(2)	23(1)
3	39(2)	67(6)	31(5)	49(4)	38(1)	48(3)
4	40(3)	57(1)	39(2)	70(6)	37(4)	50(5)
5	15(4)	15(3)	11(1)	9(2)	18(5)	17(6)
6	16(1)	15(2)	14(6)	12(5)	19(3)	22(4)