UCLA STAT 13

Introduction to Statistical Methods for the Life and Health Sciences

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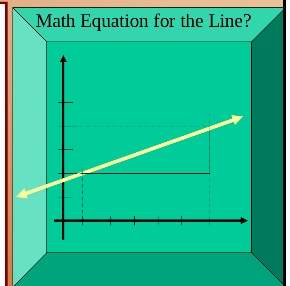
University of California, Los Angeles, Fall 2004

http://www.stat.ucla.edu/~dinov/courses_students.html

Chapter 12: Lines in 2D

(Regression and Correlation)

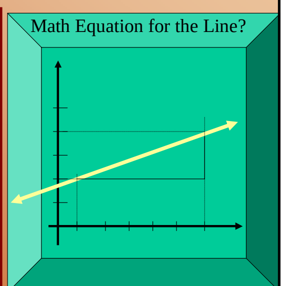
- Vertical Lines
- Horizontal Lines
- Oblique lines
- Increasing/Decreasing
- Slope of a line
- Intercept
- $Y = \alpha X + \beta$, in general.



Chapter 12: Lines in 2D

(Regression and Correlation)

- Draw the following lines:
- Y=2X+1
- Y=-3X-5
- Line through (X_1,Y_1) and (X_2,Y_2).
- $(Y-Y_1)/(Y_2-Y_1)=$
  $(X-X_1)/(X_2-X_1)$.



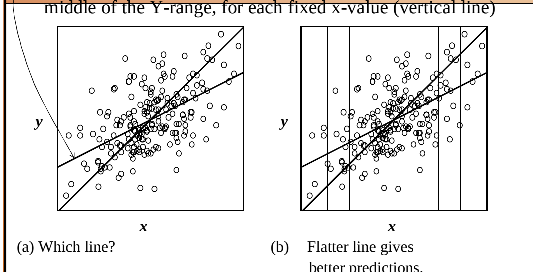
Approaches for modeling data relationships

Regression and Correlation

- There are random and nonrandom variables
- Correlation applies if both variables (X/Y) are random (e.g., We saw a previous example, systolic vs. diastolic blood pressure SISVOL/DIAVOL) and are treated symmetrically.
- Regression applies in the case when you want to single out one of the variables (response variable, Y) and use the other variable as predictor (explanatory variable, X), which explains the behavior of the response variable, Y.

Looking vertically

Flatter line gives better prediction, since it approx. goes through the middle of the Y-range, for each fixed x-value (vertical line)



(a) Which line? (b) Flatter line gives better predictions.

Figure 3.1.8 Educating the eye to look vertically.

Correlation Coefficient

Correlation coefficient ($-1 \le R \le 1$): a measure of linear association, or clustering around a line of multivariate data.

Relationship between two variables (X, Y) can be summarized by: (μ_X, σ_X), (μ_Y, σ_Y) and the correlation coefficient, R. $R=1$, perfect positive correlation (straight line relationship), $R=0$, no correlation (random cloud scatter), $R=-1$, perfect negative correlation.

Computing $R(X,Y)$: (standardize, multiply, average)

$$R(X,Y) = \frac{1}{N-1}\sum_{k=1}^{N}\left(\frac{x_k-\mu_x}{\sigma_x}\right)\left(\frac{y_k-\mu_y}{\sigma_y}\right)$$

$X=\{x_1, x_2,\ldots, x_N,\}$
$Y=\{y_1, y_2,\ldots, y_N,\}$
(μ_X, σ_X), (μ_Y, σ_Y)
sample mean / SD.

Correlation Coefficient

Example:

$$R(X,Y)=\frac{1}{N-1}\sum_{k=1}^{N}\left(\frac{x_k-\mu_x}{\sigma_x}\right)\left(\frac{y_k-\mu_y}{\sigma_y}\right)$$

Student i	Height x_i	Weight y_i	$x_i-\bar{x}$	$y_i-\bar{y}$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
1	167	60	6	4.67	36	21.8089	28.02
2	170	64	9	8.67	81	75.1689	78.03
3	160	57	-1	1.67	1	2.7889	-1.67
4	152	46	-9	-9.33	81	87.0489	83.97
5	157	55	-4	-0.33	16	0.1089	1.32
6	160	50	-1	-5.33	1	28.4089	5.33
Total	966	332	0	≈ 0	216	215.3334	195.0

Correlation Coefficient

Example:

$$R(X,Y)=\frac{1}{N-1}\sum_{k=1}^{N}\left(\frac{x_k-\mu_x}{\sigma_x}\right)\left(\frac{y_k-\mu_y}{\sigma_y}\right)$$

$$\mu_x=\frac{966}{6}=161\,\text{cm},\qquad \mu_y=\frac{332}{6}=55\,\text{kg},$$

$$\sigma_x=\sqrt{\frac{216}{5}}=6.573,\qquad \sigma_y=\sqrt{\frac{215.3}{5}}=6.563,$$

$$Corr(X,Y)=R(X,Y)=0.904$$

Correlation Coefficient - Properties

Correlation is invariant w.r.t. linear transformations of X or Y

$$R(X,Y)=\frac{1}{N-1}\sum_{k=1}^{N}\left(\frac{x_k-\mu_x}{\sigma_x}\right)\left(\frac{y_k-\mu_y}{\sigma_y}\right)=$$

$$R(aX+b,cY+d),\qquad \text{since}$$

$$\left(\frac{ax_k+b-\mu_{ax+b}}{\sigma_{ax+b}}\right)=\left(\frac{ax_k+b-(a\mu_x+b)}{|a|\times\sigma_x}\right)=$$

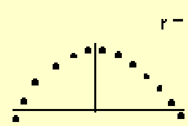
$$\left(\frac{a(x_k-\mu)+b-b}{a\times\sigma_x}\right)=\left(\frac{x_k-\mu_x}{\sigma_x}\right)$$

Correlation Coefficient - Properties

Correlation is Associative

$$R(X,Y)=\frac{1}{N}\sum_{k=1}^{N}\left(\frac{x_k-\mu_x}{\sigma_x}\right)\left(\frac{y_k-\mu_y}{\sigma_y}\right)=R(Y,X)$$

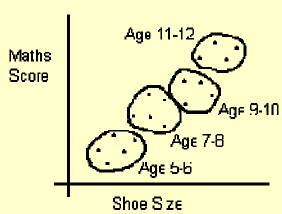
Correlation measures linear association, NOT an association in general!!! So, Corr(X,Y) could be misleading for X & Y related in a non-linear fashion.



Correlation Coefficient - Properties

$$R(X,Y)=\frac{1}{N}\sum_{k=1}^{N}\left(\frac{x_k-\mu_x}{\sigma_x}\right)\left(\frac{y_k-\mu_y}{\sigma_y}\right)=R(Y,X)$$

1. *R* measures the extent of linear association between two continuous variables.
2. Association does not imply causation - both variables may be affected by a third variable – age was a confounding variable.

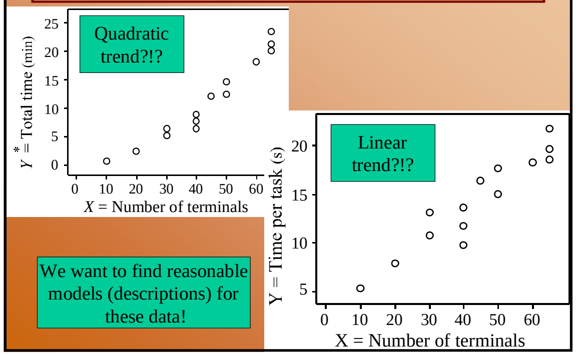


Trend and Scatter - Computer timing data

- The major components of a regression relationship are trend and scatter around the trend.
- To investigate a trend – fit a math function to data, or smooth the data.
- Computer timing data: a mainframe computer has X users, each running jobs taking Y min time. The main CPU swaps between all tasks. Y* is the total time to finish all tasks. Both Y and Y* increase with increase of tasks/users, but how?

X	=	Number of terminals:	40	50	60	45	40	10	30	20
Y*	=	Total Time (mins):	6.6	14.9	18.4	12.4	7.9	0.9	5.5	2.7
Y	=	Time Per Task (secs):	9.9	17.8	18.4	16.5	11.9	5.5	11	8.1
X	=	Number of terminals:	50	30	65	40	65	65		
Y*	=	Total Time (mins):	12.6	6.7	23.6	9.2	20.2	21.4		
Y	=	Time Per Task (secs):	15.1	13.3	21.8	13.8	18.6	19.8		

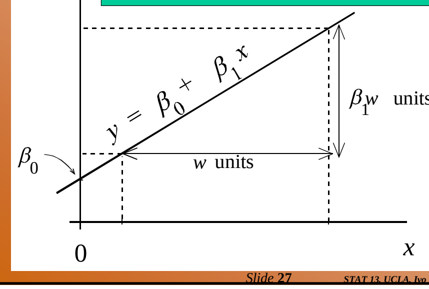
Trend and Scatter - Computer timing data



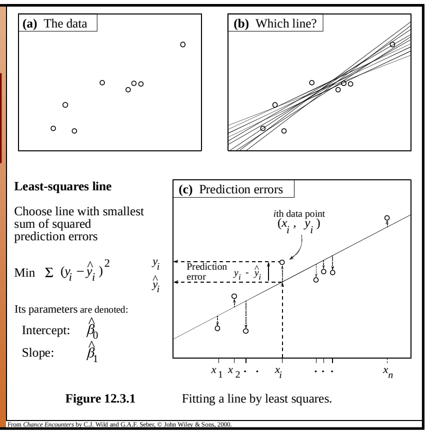
We want to find reasonable models (descriptions) for these data!

Equation for the straight line – linear/affine function

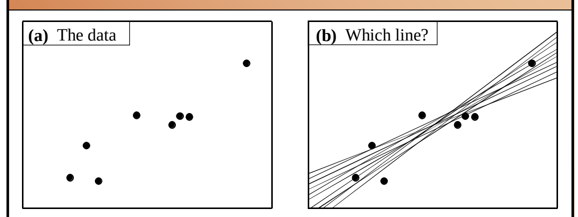
β_0=Intercept (the y-value at x=0)
β_1=Slope of the line (rise/run), change of y for every unit of increase for x.



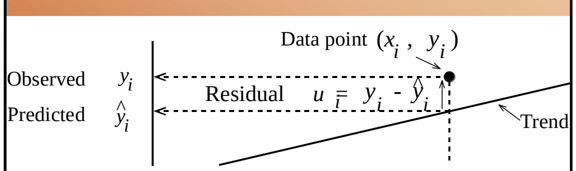
Choosing the “best-fitting” line



Fitting a line through the data



The idea of a residual or prediction error



Least squares criterion

Least squares criterion: Choose the values of the parameters to *minimize the sum of squared prediction errors* (or sum of squared residuals),

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

The least squares line

Least-squares line

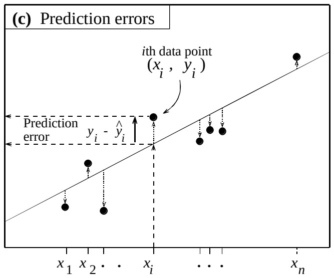
Choose line with smallest sum of squared prediction errors

Min $\Sigma\ (y_i - \hat{y}_i)^2$

Its parameters are denoted:

Intercept: $\hat{\beta}_0$

Slope: $\hat{\beta}_1$



Least-squares line: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

The least squares line

Least-squares line: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}\left[(x_i - \bar{x})(y_i - \bar{y})\right]}{\sum_{i=1}^{n}(x_i - \bar{x})^2}; \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

Computer timings data – linear fit

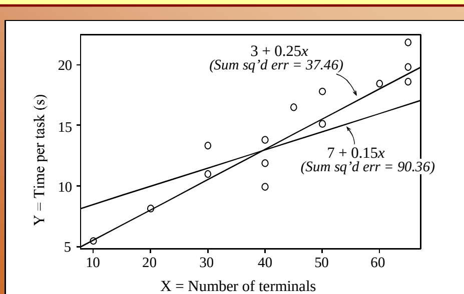


Figure 12.3.2 Two lines on the computer-timings data.

Computer timings data

TABLE 12.3.1 Prediction Errors

		3 + 0.25x		7 + 0.15x	
x	y	$\hat{y}$	$y-\hat{y}$	$\hat{y}$	$y-\hat{y}$
40	9.90	13.00	-3.10	13.00	-3.10
50	17.80	15.50	2.30	14.50	3.30
60	18.40	18.00	0.40	16.00	2.40
45	16.50	14.25	2.25	13.75	2.75
40	11.90	13.00	-1.10	13.00	-1.10
10	5.50	5.50	0.00	8.50	-3.00
30	11.00	10.50	0.50	11.50	-0.50
20	8.10	8.00	0.10	10.00	-1.90
50	15.10	15.50	-0.40	14.50	0.60
30	13.30	10.50	2.80	11.50	1.80
65	21.80	19.25	2.55	16.75	5.05
40	13.80	13.00	0.80	13.00	0.80
65	18.60	19.25	-0.65	16.75	1.85
65	19.80	19.25	0.55	16.75	3.05
	Sum of squared errors		37.46		90.36

Adding the least squares line

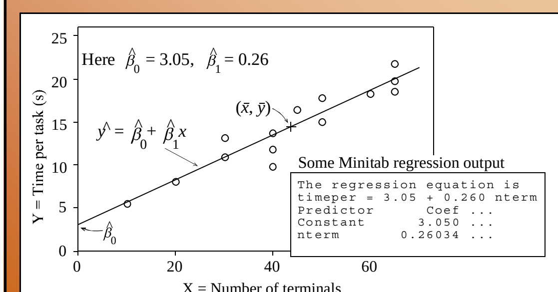


Figure 12.3.3 Computer-timings data with least-squares line.

Review, Fri., Oct. 19, 2001

1. The least-squares line $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ passes through the points (x = 0, $\hat{y}$ = ?) and (x = $\bar{x}$, $\hat{y}$ = ?). Supply the missing values.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}\left[(x_i - \bar{x})(y_i - \bar{y})\right]}{\sum_{i=1}^{n}(x_i - \bar{x})^2}; \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

Hands – on worksheet !

1. X={-1, 2, 3, 4}, Y={0, -1, 1, 2},

X	Y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})\times(y-\bar{y})$	
-1	0						
2	-1						
3	1						
4	2						

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}\left[(x_i-\bar{x})(y_i-\bar{y})\right]}{\sum_{i=1}^{n}(x_i-\bar{x})^2}; \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

Hands – on worksheet !

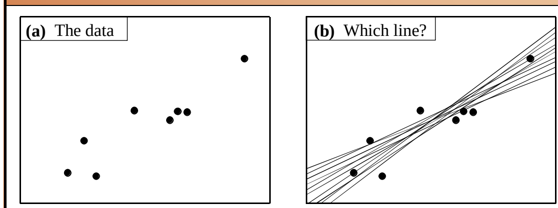
1. X={-1, 2, 3, 4}, Y={0, -1, 1, 2}, $\bar{x} = 2$, $\bar{y} = 0.5$

X	Y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})\times(y-\bar{y})$	
-1	0	-3	-0.5	9	0.25	1.5	
2	-1	0	-1.5	0	2.25	0	
3	1	1	0.5	1	0.25	0.5	
4	2	2	1.5	4	2.25	3	
2	0.5			14	5	5	

β_1=5/14
β_0=y^-β1*x^
β_0= 0.5-10/14

Fitting a line through the data

Show the Regression-Line Simulation Applet
RegressionApplet.html



Fitting a line through the data

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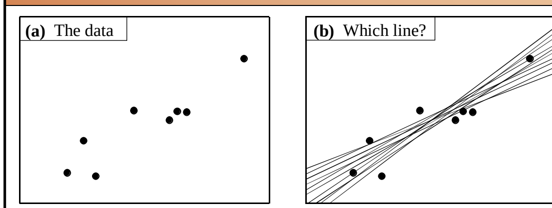
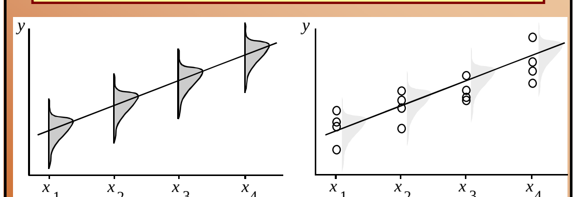


The simple linear model



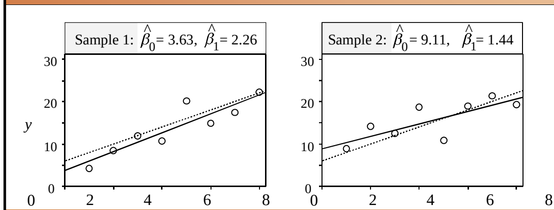
(a) The simple linear model (b) Data sampled from the model

When $X = x$, $Y \sim \text{Normal}(\mu_Y, \sigma)$ where $\mu_Y = \beta_0 + \beta_1 x$, **OR**

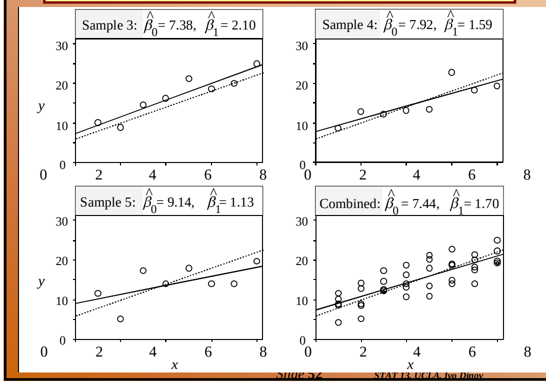
when $X = x$, $\boldsymbol{Y = \beta_0 + \beta_1 x + U}$, where $U \sim \text{Normal}(0, \sigma)$

Random error

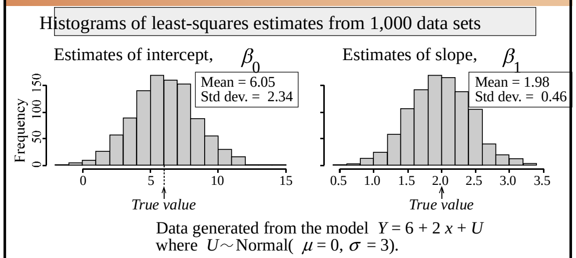
Data generated from $Y = 6 + 2x$ + error (U)
Dotted line ············ is true line and
solid line ——— is the data-estimated LS line.
Note differences between true β_0=6, β_1=2 and their estimates β_0^ & β_1^.



Data generated from $Y = 6 + 2x$ + error(U)



Data generated from $Y = 6 + 2x$ + error(U)



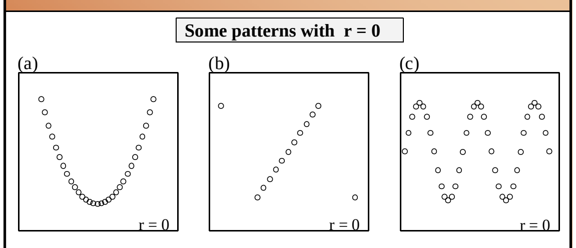
Recall the correlation coefficient…

Another form for the correlation coefficient is:

$$R(X;Y) = Corr(X;Y) = \frac{\sum_{i=1}^{n}\left[(x_i-\bar{x})(y_i-\bar{y})\right]}{\sqrt{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\times\left(\sum_{i=1}^{n}(y_i-\bar{y})^2\right)}} =$$

$$\frac{\sum_{i=1}^{n}\left[y_i x_i + \bar{x}\bar{y} - \bar{y}x_i - \bar{x}y_i\right]}{\sqrt{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\left(\sum_{i=1}^{n}(y_i-\bar{y})^2\right)}} = \frac{\sum_{i=1}^{n}\left(y_i x_i\right) - n\times\bar{x}\times\bar{y}}{\sqrt{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\times\left(\sum_{i=1}^{n}(y_i-\bar{y})^2\right)}}$$

Misuse of the correlation coefficient

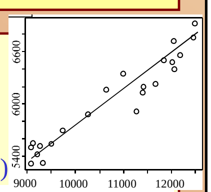


From *Chance Encounters* by C.J. Wild and G.A.F. Seber, © John Wiley & Sons, 2000.

Linear Regression

- Regression relationship = trend + residual scatter

$$\hat{y} = \underbrace{\hat\beta_0 + \hat\beta_1 x} + \underbrace{Err}$$



- Trend=best linear fit Line (LS)

$$\hat\beta_1 = \sum_{i=1}^{n}\left[(x_i-\bar{x})(y_i-\bar{y})\right] \Big/ \sum_{i=1}^{n}(x_i-\bar{x})^2; \quad \hat\beta_0 = \bar{y} - \hat\beta_1\bar{x}$$

- Scatter = residual (prediction) error **Err=Obs-Pred**

$$\sum_{i=1}^{n}(y_i-\hat{y}_i)^2 = (y_1-\hat{y}_1)^2 + (y_2-\hat{y}_2)^2 + \ldots + (y_n-\hat{y}_n)^2$$

Another Notation for the Slope of the LS line

1. Note that there is a slight difference in the formula for the slope of the Least-Squares Best-Linear Fit line:

$$\hat\beta_1 = \frac{\sum_{i=1}^{n}\left[(x_i-\bar{x})(y_i-\bar{y})\right]}{\sum_{i=1}^{n}(x_i-\bar{x})^2}; \quad \hat\beta_0 = \bar{y} - \hat\beta_1\bar{x}$$

$$\hat\beta_1 = Corr(X;Y)\times\frac{SD(Y)}{SD(X)}; \quad \hat\beta_0 = \bar{y} - \hat\beta_1\bar{x}$$

Another Notation for the Slope of the LS line

$$\boxed{\hat{\beta}_1^{\,New} = Corr(X;Y)\times\frac{SD(Y)}{SD(X)}} =$$

$$\frac{\sum_{i=1}^{n}\left[(x_i-\bar{x})(y_i-\bar{y})\right]}{\sqrt{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\times\left(\sum_{i=1}^{n}(y_i-\bar{y})^2\right)}}\times\frac{\sqrt{\left(\sum_{i=1}^{n}(y_i-\bar{y})^2\right)\frac{1}{N-1}}}{\sqrt{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\frac{1}{N-1}}} =$$

$$\boxed{\frac{\sum_{i=1}^{n}\left[(x_i-\bar{x})(y_i-\bar{y})\right]}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \hat{\beta}_1^{\,Old}}$$

Course Material Review

1. ===========Part I=================
2. Data collection, surveys.
3. Experimental vs. observational studies
4. Numerical Summaries (5-#-summary)
5. Binomial distribution (prob's, mean, variance)
6. Probabilities & proportions, independence of events and conditional probabilities
7. Normal Distribution and normal approximation

Course Material Review – cont.

1. ===============Part II=================
2. Central Limit Theorem – sampling distribution of $\overline{X}$
3. Confidence intervals and parameter estimation
4. Hypothesis testing
5. Paired vs. Independent samples
6. Chi-Square (χ^2) Goodness-of-fit Test
7. Analysis Of Variance (1-way-ANOVA, one categorical var.)
8. Correlation and regression
9. Best-linear-fit, least squares method