UCLA STAT 35
Applied Computational and Interactive Probability

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Probability

Sample Spaces and Events

Sample Space

The **sample space** of an experiment, denoted S , is the set of all possible outcomes of that experiment.

Sample Space

Ex. Roll a die 

Outcomes: landing with a 1, 2, 3, 4, 5, or 6 face up.

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

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DiceSampleExperiment.html

DiceExperiment.html

Events

An **event** is any collection (subset) of outcomes contained in the sample space S. An event is **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

Relations from Set Theory

1. The **union** of two events A and B is the event consisting of all outcomes that are either in A or in B.

   Notation: $A \cup B$

   Read: A or B

Relations from Set Theory

2. The **intersection** of two events A and B is the event consisting of all outcomes that are in both A and B.

Notation: $A \cap B$

Read: A and B

Relations from Set Theory

3. The **complement** of an event A is the set of all outcomes in S that are not contained in A.

Notation: A'

Events

Ex. Rolling a die. $S = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

$A \cup B = \{1, 2, 3, 5\}$

$A \cap B = \{1, 3\}$

$A' = \{4, 5, 6\}$

Mutually Exclusive

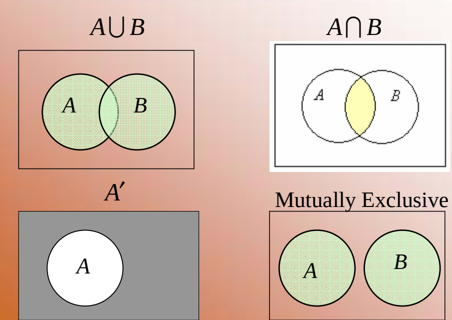
When A and B have no outcomes in common, they are **mutually exclusive** or **disjoint** events

Mutually Exclusive

Ex. When rolling a die, if event $A = \{2, 4, 6\}$ (evens) and event $B = \{1, 3, 5\}$ (odds), then A and B are mutually exclusive.

Ex. When drawing a single card from a standard deck of cards, if event A = {heart, diamond} (red) and event B = {spade, club} (black), then A and B are mutually exclusive.

Venn Diagrams



Axioms, Interpretations, and Properties of Probability

Axioms of Probability

Axiom 1 $P(A) \geq 0$ for any event A

Axiom 2 $P(S) = 1$

If all A_i's are mutually exclusive, then

Axiom 3 $P(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{i=1}^{k} P(A_i)$ (finite set)

$$P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

(infinite set)

Properties of Probability

For any event A, $P(A) = 1 - P(A')$.

If A and B are mutually exclusive, then $P(A \cap B) = 0$.

For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Ex. A card is drawn from a well-shuffled deck of 52 playing cards. What is the probability that it is a queen or a heart?

Q = Queen and H = Heart

$$P(Q) = \frac{4}{52},\ P(H) = \frac{13}{52},\ P(Q \cap H) = \frac{1}{52}$$

$$P(Q \cup H) = P(Q) + P(H) - P(Q \cap H)$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \quad = \frac{16}{52} = \frac{4}{13}$$

Counting Techniques

Product Rule

If the first element or object of an ordered pair can be used in n_1 ways, and for each of these n_1 ways the second can be selected n_2 ways, then the number of pairs is $n_1 n_2$.

** Note that this generalizes to k elements (k – tuples)

Permutations

Any ordered sequence of k objects taken from a set of n distinct objects is called a **permutation** of size k of the objects.

MatchExperiment.html

Notation: $P_{k,n}$

$$P_{k,n} = n(n-1)\cdot\ldots\cdot(n-k+1)$$

Factorial

For any positive integer m, $m!$ is read "m factorial" and is defined by $m! = m(m-1)\cdot\ldots\cdot(2)(1)$. Also, $0! = 1$.

Note, now we can write:

$$P_{k,n} = \frac{n!}{(n-k)!}$$

Ex. A boy has 4 beads – red, white, blue, and yellow. How different ways can three of the beads be strung together in a row?

This is a permutation since the beads will be in a row (order).

$$P_{3,4} = \frac{4!}{(4-3)!} = 4! = 24$$

number selected | total

24 different ways

Combinations

Given a set of n distinct objects, any unordered subset of size k of the objects is called a **combination**.

Notation: $\binom{n}{k}$ or $C_{k,n}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Ex. A boy has 4 beads – red, white, blue, and yellow. How different ways can three of the beads be chosen to trade away?

This is a combination since they are chosen without regard to order.

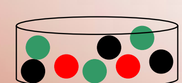
$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4$$

total | number selected

4 different ways

Ex. Three balls are selected at random without replacement from the jar below. Find the probability that one ball is red and two are black.

UrnExperiment.html



$$= \frac{\binom{2}{1}\cdot\binom{3}{2}}{\binom{8}{3}} = \frac{2\cdot 3}{56} = \frac{3}{28}$$

Conditional Probability

Conditional Probability

For any two events A and B with $P(B) > 0$, the **conditional probability** of A given that B has occurred is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Which can be written:

$$P(A \cap B) = P(B) \cdot P(A \mid B)$$

The Law of Total Probability

If the events $A_1, A_2, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B:

$$P(B) = \sum_{i=1}^{k} P\left(B \bigcap A_i\right) = \sum_{i=1}^{k} P(B \mid A_i) P(A_i)$$

Bayes' Theorem

Let $A_1, A_2, \ldots, A_n$ be a collection of k mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, 2, \ldots, k$. Then for any other events B & C for which $P(B) > 0$

$$P(C \mid B) = \frac{P(\mathbf{B} \mid C) \times P(C)}{\sum_{k=1}^{n} P(B \mid A_k) P(A_k)}$$

$$j = 1, 2 \ldots, k$$

Ex. A store stocks light bulbs from three suppliers. Suppliers A, B, and C supply 10%, 20%, and 70% of the bulbs respectively. It has been determined that company A's bulbs are 1% defective while company B's are 3% defective and company C's are 4% defective. If a bulb is selected at random and found to be defective, what is the probability that it came from supplier B?

Let D = defective

$$P(B \mid D) = \frac{P(B)P(D \mid B)}{P(A)P(D \mid A) + P(B)P(D \mid B) + P(C)P(D \mid C)}$$

$$= \frac{0.2(0.03)}{0.1(0.01) + 0.2(0.03) + 0.7(0.04)} \approx 0.1714$$

So about 0.17

Bayesian Rule

- If $\{A_1, A_2, \ldots, A_n\}$ are a non-trivial partition of the sample space (mutually exclusive and $\cup A_i = S$, $P(A_i) > 0$) *then for any non-trivial event* and B ($P(B) > 0$)

$$\mathbf{P(A_i \mid B) = P(A_i \cap B) / P(B) = [P(B \mid A_i) \times P(A_i)] / P(B)} =$$

$$= \frac{P(B \mid A_i) \times P(A_i)}{\sum_{k=1}^{n} P(B \mid A_k) P(A_k)}$$

Bayesian Rule

$$P(C \mid B) = \frac{P(\mathbf{B} \mid C) \times P(C)}{\sum_{k=1}^{n} P(B \mid A_k) P(A_k)}$$

D **= the test person has the disease.**

T **= the test result is positive.**

Ex: (Laboratory blood test) **Assume:**

P(positive Test| Disease) = 0.95

P(positive Test| no Disease)=0.01

P(Disease) = 0.005

Find:

P(Disease | positive Test)=?

P(D | T) = ?

$$P(D \mid T) = \frac{P(D \cap T)}{P(T)} = \frac{P(T \mid D) \times P(D)}{P(T \mid D) \times P(D) + P(T \mid D^c) \times P(D^c)}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} = \frac{0.00475}{0.0147} = 0.323$$

Classes vs. Evidence Conditioning

- **Classes**: healthy(NC), cancer
- **Evidence**: positive mammogram (pos), negative mammogram (neg)
- If a woman has a positive mammogram result, what is the probability that she has breast cancer?

$$P(class \mid evidence) = \frac{P(evidence \mid class) \times P(class)}{\sum_{classes} P(evidence \mid class) \times P(class)}$$

$$P(cancer) = 0.01$$

$$P(pos \mid cancer) = 0.8$$

$$P(pos \mid healthy) = 0.1$$

$$P(cancer \mid pos) = ?$$

P(C|P)=P(P|C)xP(C)/[P(P|C)xP(C)+ P(P|H)xP(H)]

P(C|P)=0.8x0.01 / [0.8x0.01 + 0.1x0.99] = ?

Independence

Independent Events

Two event *A* and *B* are **independent events** if $P(A \mid B) = P(A)$.

Otherwise *A* and *B* are dependent.

Independent Events

Events *A* and *B* are independent events if and only if

$$P(A \cap B) = P(A)P(B)$$

** Note: this generalizes to more than two independent events.

Theory of Counting = Combinatorial Analysis

Generalized Principle of Counting: If M (independent) experiments are performed and the first one has N_m possible outcomes, 1<=m<=M, then the TOTAL number of outcomes of the combined experiment is

$$N_1 x N_2 x \ldots x N_M.$$

E.g., How many binary functions [f(i)=0 or f(i)=1], defined on a grid 1, 2, 3, …, n, are there? How many numbers can be stored in 8 bits = 1 byte?

$2 \times 2 \times \ldots \times 2 = 2^n$

Permutation & Combination

Permutation: Number of **ordered** arrangements of **r** objects chosen from **n** ***distinctive*** objects

$$P_n^r = n(n-1)(n-2)\ldots(n-r+1)$$

$$P_n^n = P_n^{n-r} \cdot P_r^r$$

e.g. $P_6^3 = 6 \cdot 5 \cdot 4 = 120.$

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Permutation & Combination

Combination: Number of **non-ordered** arrangements of r objects chosen from n ***distinctive*** objects:

$$C_n^r = P_n^r / r! = \frac{n!}{(n-r)!r!}$$

Or use notation of $\binom{n}{r} = C_n^r$

e.g. 3!=6 , 5!=120 , 0!=1

$$\binom{7}{3} = \frac{7!}{4!3!} = 35$$

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Permutation & Combination

Combinatorial Identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Analytic proof: (expand both hand sides)

Combinatorial argument: Given n object focus on one of them (obj. 1). There are $\binom{n-1}{r-1}$ groups of size r that contain obj. 1 (since each group contains r-1 other elements out of n-1). Also, there are $\binom{n-1}{r}$ groups of size r, that do not contain obj1. But the total of all r-size groups of n-objects is $\binom{n}{r}$!

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Permutation & Combination

Combinatorial Identity:

$$\binom{n}{r} = \binom{n}{n-r}$$

Analytic proof: (expand both hand sides)

Combinatorial argument: Given n objects the number of combinations of choosing any r of them is equivalent to choosing the remaining n-r of them (order-of-objs-not-important!)

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Examples

1. Suppose car plates are 7-digit, like AB1234. If all the letters can be used in the first 2 places, and all numbers can be used in the last 4, how many different plates can be made? How many plates are there with no repeating digits?

Solution: a) $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

b) $P_{26}^2 \cdot P_{10}^3 = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

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Examples

2. How many different letter arrangement can be made from the 11 letters of MISSISSIPPI?

Solution: There are: 1 M, 4 I, 4 S, 2 P letters.

Method 1: consider different permutations:

11!/(1!4!4!2!)=34650

Method 2: consider combinations:

$$\binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2} = \ldots = \binom{11}{2}\binom{9}{4}\binom{5}{4}\binom{1}{1}$$

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Examples

3. There are N telephones, and any 2 phones are connected by 1 line. Then how many lines are needed all together?

Solution: $C^2_N = N(N - 1)/2$
If, N=5, complete graph with 5 nodes has C^2_5=10 edges.

Examples

4. **N distinct balls** with **M of them white**. Randomly **choose n of the N balls**. What is the probability that the sample contains exactly **m white balls** (suppose every ball is equally likely to be selected)?

Solution: a) For the event to occur, **m out of M white** balls are chosen, and **n-m out of N-M non-white** balls are chosen. And we get $\binom{M}{m}\binom{N-M}{n-m}$

b) Then the probability is $\binom{M}{m}\binom{N-M}{n-m}/\binom{N}{n}$

These Probabilities Are associated with the name HyperGeometric(N, n, M) distrib.

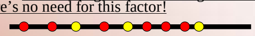
Examples

5. N boys (●) and M girls (○), M<=N+1, stand in 1 line. How many arrangements are there so that no 2 girls stand next to each other?

There are **N!** ways of ordering the boys among themselves

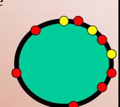
There are **M!** ways of ordering the girls among themselves. NOTE – if girls are indistinguishable then there's no need for this factor!

Solution: $N!\cdot\binom{N+1}{M}\cdot M!$



There are **N+1** slots for the girls to fill between the boys And there are **M** girls to position in these slots, hence the coefficient in the middle.

How about they are arranged in a circle?
Answer: $N!\binom{N}{M}M!$



E.g., N=3, M=2

Examples

5a. How would this change if there are N functional (●) and M defective chips (○), M<=N+1, in an assembly line?

Solution: $\binom{N+1}{M}$

There are **N+1** slots for the girls to fill between the boys And there are **M** girls to position in these slots, hence the coefficient in the middle.

Binomial theorem & multinomial theorem

Binomial theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Deriving from this, we can get such useful formula (a=b=1)

$$\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}=2^n=(1+1)^n$$

Also from $(1+x)^{m+n}=(1+x)^m(1+x)^n$ we obtain:

$$\binom{m+n}{k}=\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$$

On the left is the coeff of $1^k x^{(m+n-k)}$. On the right is the same coeff in the product of $(\ldots+ \text{coeff} * x^{(m-i)} +\ldots) * (\ldots+\text{coeff} * x^{(n-k+i)} +\ldots)$.

Multinomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Generalization: Divide n distinctive objects into k groups, with the size of every group $n_1,\ldots,n_k$, and $n_1+n_2+\ldots+n_k = n$

$$(x_1+x_2+\ldots+x_k)^n = \sum \binom{n}{n_1,n_2,\ldots,n_k} x_1^{n_1} x_2^{n_2}\ldots x_k^{n_k}$$

where $\binom{n}{n_1,n_2,\ldots,n_k}=\binom{n}{n_1}\binom{n-n_1}{n_2}\ldots\binom{n-n_1-\ldots-n_{k-1}}{n_k}=\frac{n!}{n_1!n_2!\ldots n_k!}$

Multinomial Probabilities $p(n_1,\ldots,n_k)=\frac{n!}{n_1!\cdots n_k!}p_1^{n_1}\cdots p_k^{n_k}$

Multinomial theorem

- N independent trials with results falling in one of k possible categories labeled 1, …, k. Let p_i = the probability of a trial resulting in the i^{th} category, where $p_1+\ldots+p_k=1$
- N_i = number of trials resulting in the i^{th} category, where $N_1+\ldots+N_k = N$
- Ex: Suppose we have 9 people arriving at a meeting.

  P(by Air) = 0.4, *P*(by Bus) = 0.2

  P(by Automobile) = 0.3, *P*(by Train) = 0.1

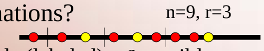
  P(3 by Air, 3 by Bus, 1 by Auto, 2 by Train) = ?

  P(2 by air) = ?

Examples

7. There are **n balls** randomly positioned in **r distinguishable urns**. Assume n>= r. What is the number of possible combinations?

n=9, r=3



1) If the balls are distinguishable (labeled) : r^n possible outcomes, where empty urns are permitted. Since each of the ***n*** balls can be placed in any of the ***r*** urns.

2) If the balls are indistinguishable: **no empty urns** are allowed – select r-1 of all possible n-1 dividing points between the n-balls. $\binom{n-1}{r-1}$

3) If **empty urns** are allowed

n=9, r=3, and ○ are empty bins

$$\binom{n+r-1}{r-1}$$

Application – Number of integer solutions to linear equ's

1) There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying

$$x_1+ x_2 + \ldots + x_r = n, \ \& \ x_i >0, 1<=i<=r$$

2) There are $\binom{n+r-1}{r-1}$ distinct positive integer-valued vectors $(y_1, y_2, \ldots, y_r)$ satisfying

$$y_1+ y_2 + \ldots + y_r = n, \ \& \ y_i >=0, 1<=i<=r$$

Since there are *n+r-1* possible positions for the dividing splitters (or by letting $y_i=x_i-1$, RHS=n+r).

Example

1) An investor has \$20k to invest in 4 potential stocks. Each investment is in increments of \$1k, to minimize transaction fees. In how many different ways can the money be invested?

2) $x_1+x_2+x_3+x_4=20, x_k>=0$ ➔ $\binom{23}{3}=1{,}771$

3) If not all the money needs to be invested, let x5 be the left over money, then

$x_1+x_2+x_3+x_4 + x_5 =20$ $\binom{24}{4}=10{,}626$

Examples

8. Randomly give n pairs of **distinctive shoes** to n people, with 2 shoes to everyone. How many arrangements can be made? How many arrangements are there, so that everyone gets an original pair? What is the the probability of the latter event, ***E***?

Solution: a) according to $\binom{2n}{n_1, n_2, \ldots, n_r} =$

Note: **r = n = # of pairs**!

total arrangements is $\binom{2n}{n_1}\binom{2n-n_1}{n_2}\ldots\binom{2n-n_1-n_2-\ldots-n_{r-1}}{n_r} =$

N=(2n)!/(2!)^r = (2n)!/2^r $\qquad \dfrac{(2n)!}{n_1!n_2!\ldots n_r!} = \dfrac{(2n)!}{(2!)^r}$

b) Regard every shoe pair as one object, and give them to people, there are M=n! arrangements.

c)P(***E***)=M/N= n! /[(2n)!/2^n]=1/(2n-1)!! (Do n=6, case by hand!)

*note: n!!=n(n-2)(n-4)…

Sterling Formula for asymptotic behavior of n!

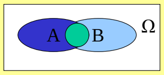
Sterling formula:

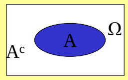
$$n! \sim \sqrt{\frac{2\pi}{n}} \times \left(\frac{n}{e}\right)^n$$

$$\frac{n!}{\sqrt{\frac{2\pi}{n}} \times \left(\frac{n}{e}\right)^n} \xrightarrow[n \longrightarrow \infty]{} 1$$

Probability and Venn diagrams

Venn's diagram





Union: A∪B

Intersection: A∩B

A^c denotes the part in Ω but not in A.

Properties:

A∩B= B∩A,	*AUB= BUA,*
(A∩B)∩C=A∩(B∩C),	*(A U B) UC=A U (B U C),*
(A∩B)UC=(AUC)∩(BUC)	*(A U B) ∩ C=(A ∩ C) U (B ∩ C)*

De Morgan's Law: $A^c \cap B^c = (A \cup B)^c$, $A^c \cup B^c = (A \cap B)^c$

Generalized: $(\cap E_i)^c = \cup E_i^c$, $(\cup E_i)^c = \cap E_i^c$, $i = 1,2,\ldots,n$

Probability and Venn diagrams

Proposition

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) =$$

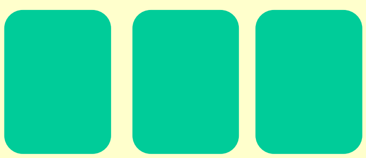
$$\sum_{i=1}^{n} P(A_i) - \sum_{1 \le i1 < i2 \le n} P\left(A_{i1} \bigcap A_{i2}\right) + \ldots$$

$$+ (-1)^{r+1} \sum_{1 \le i1 < j2 < \ldots < ir \le n} P\left(A_{i1} \bigcap A_{i2} \bigcap \ldots \bigcap A_{ir}\right) + \ldots$$

$$+ (-1)^{n+1} P\left(A_{i1} \bigcap A_{i2} \bigcap \ldots \bigcap A_{in}\right)$$

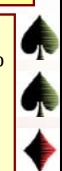
Let's Make a Deal Paradox – aka, Monty Hall 3-door problem

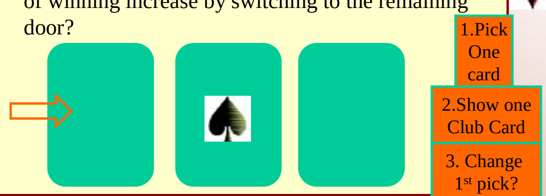
- This paradox is related to a popular television show in the 1970's. In the show, a contestant was given a choice of three doors/cards of which one contained a prize (diamond). The other two doors contained gag gifts like a chicken or a donkey (clubs).



Let's Make a Deal Paradox.

- After the contestant chose an initial door, the host of the show then revealed an empty door among the two unchosen doors, and asks the contestant if he or she would like to switch to the other unchosen door. The question is should the contestant switch. Do the odds of winning increase by switching to the remaining door?





Let's Make a Deal Paradox.

- The *intuition* of most people tells them that each of the doors, the chosen door and the unchosen door, are equally likely to contain the prize so that there is a 50-50 chance of winning with either selection? This, however, is **not the case**.
- The probability of winning by using the switching technique is 2/3, while the odds of winning by not switching is 1/3. The easiest way to explain this is as follows:
- MonteHallExperiment/Game.html