UCLA STAT 35

Applied Computational and Interactive Probability

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Statistics & Their Distributions –

The CLT

Statistic

A *statistic* is any quantity whose value can be calculated from sample data. Prior to obtaining data, there is uncertainty as to what value of any particular statistic will result. A statistic is a random variable denoted by an uppercase letter; a lowercase letter is used to represent the calculated or observed value of the statistic.

Random Samples

The rv's $X_1,\ldots,X_n$ are said to form a (simple *random sample* of size n if

1. The X_i's are independent rv's.
2. Every X_i has the same probability distribution.

Simulation Experiments

The following characteristics must be specified:

1. The statistic of interest.
2. The population distribution.
3. The sample size n.
4. The number of replications k.

The Distribution of the Sample Mean

Using the Sample Mean

Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean value μ and standard deviation σ. Then

1. $E(\overline{X}) = \mu_{\overline{X}} = \mu$
2. $V(\overline{X}) = \sigma^2_{\overline{X}} = \sigma^2 / n$

In addition, with $T_o = X_1 + \ldots + X_n$, $E(T_o) = n\mu$, $V(T_o) = n\sigma^2$, and $\sigma_{T_o} = \sqrt{n}\sigma$.

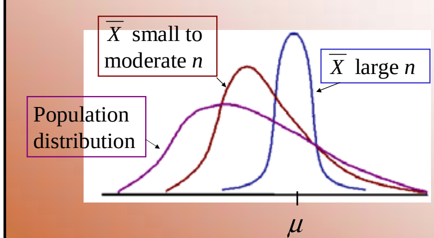
Normal Population Distribution

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean value μ and standard deviation σ. Then for any n, $\overline{X}$ is normally distributed, as is T_o.

The Central Limit Theorem

Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean value μ and variance σ^2. Then if n sufficiently large, $\overline{X}$ has approximately a normal distribution with $\mu_{\overline{X}} = \mu$ and $\sigma^2_{\overline{X}} = \sigma^2 / n$, and T_o also has approximately a normal distribution with $\mu_{T_o} = n\mu$, $\sigma_{T_o} = n\sigma^2$. The larger the value of n, the better the approximation.

The Central Limit Theorem



Rule of Thumb

If $n > 30$, the Central Limit Theorem can be used.

Approximate Lognormal Distribution

Let $X_1, \ldots, X_n$ be a random sample from a distribution for which only positive values are possible [$P(X_i > 0) = 1$]. Then if n is sufficiently large, the product $Y = X_1 X_2 \ldots X_n$ has approximately a lognormal distribution.

Central Limit Theorem – heuristic formulation

Central Limit Theorem:
When sampling from almost any distribution,
$\overline{X}$ is approximately Normally distributed in large samples.

Show Sampling Distribution Simulation Applet:
file:///C:/Ivo.dir/UCLA_Classes/Winter2002/AdditionalInstructorAids/
SamplingDistributionApplet.html

Independence

- For discrete random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are independent.

(1) $f_{XY}(x,y) = f_X(x)\, f_Y(y)$ for all x and y

(2) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$

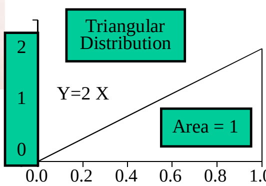
(3) $f_{X|y}(y) = f_X(x)$ for all x and y with $f_Y(y) > 0$

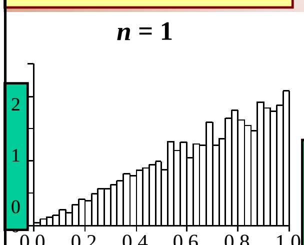
(4) $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y respectively.

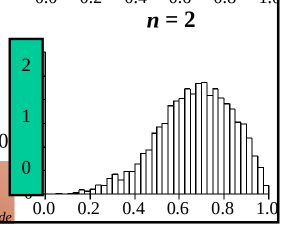
Recall we looked at the sampling distribution of $\overline{X}$

- For the sample mean calculated from a random sample, $E(\overline{X}) = \mu$ and $SD(\overline{X}) = \sigma/\sqrt{n}$, provided $\overline{X} = (X_1+X_2+ \dots + X_n)/n$, and $X_k \sim N(\mu, \sigma)$. Then
- $\overline{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$. And variability from sample to sample in the ***sample-means*** is given by the variability of the individual observations divided by the square root of the sample-size. In a way, averaging decreases variability.

Central Limit Effect – Histograms of sample means



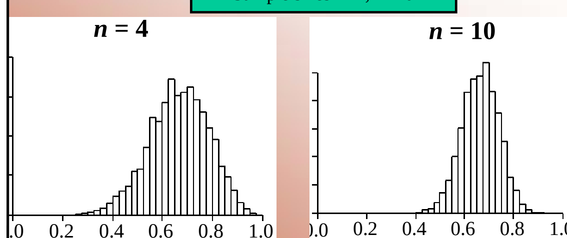


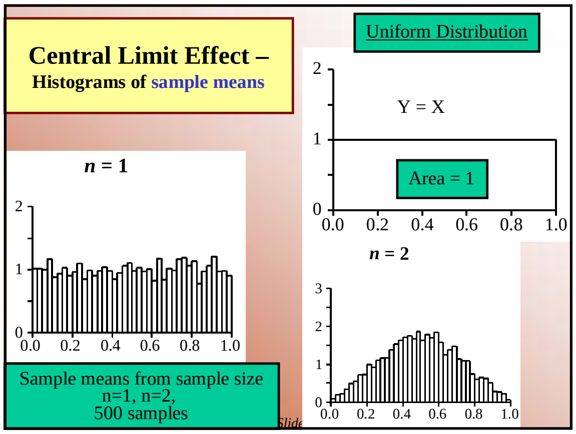


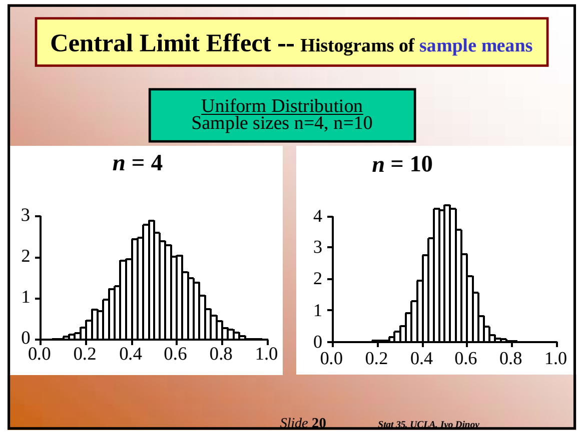
Sample means from sample size n=1, n=2, 500 samples

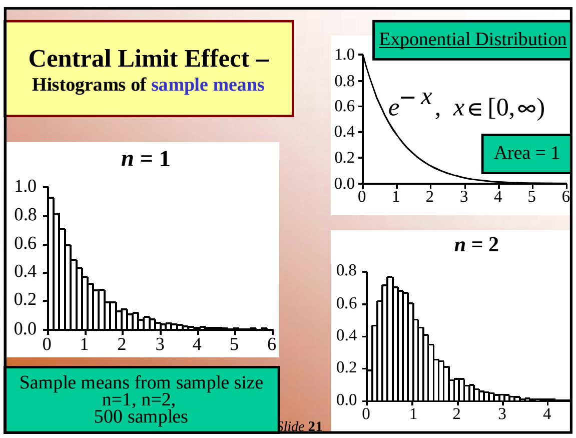
Central Limit Effect -- Histograms of sample means

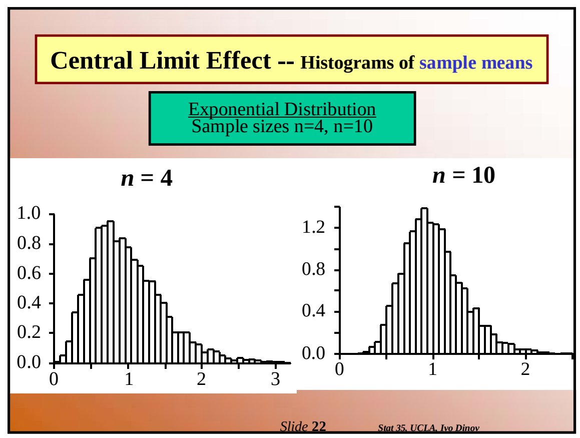
Triangular Distribution
Sample sizes n=4, n=10

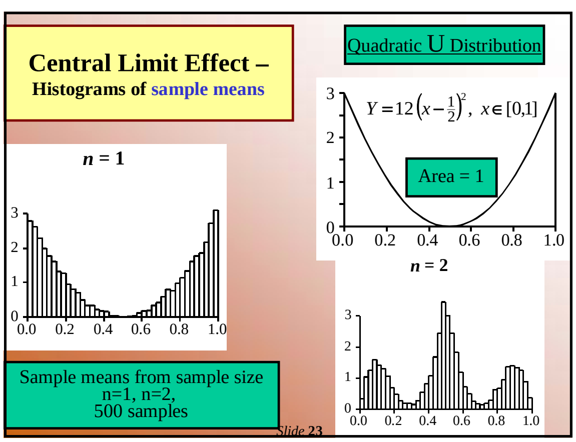


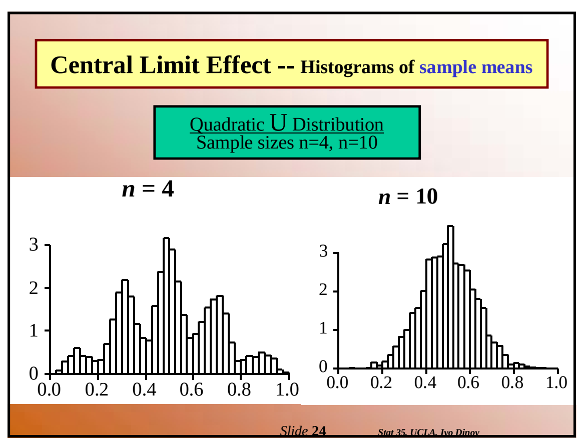

Central Limit Effect –
Histograms of sample means
Uniform Distribution
Y = X
Area = 1
n = 1
n = 2
Sample means from sample size
n=1, n=2,
500 samples


Central Limit Effect -- Histograms of sample means
Uniform Distribution
Sample sizes n=4, n=10
n = 4
n = 10
Slide 20


Central Limit Effect –
Histograms of sample means
Exponential Distribution
$e^{-x}, \; x \in [0, \infty)$
Area = 1
n = 1
n = 2
Sample means from sample size
n=1, n=2,
500 samples
Slide 21


Central Limit Effect -- Histograms of sample means
Exponential Distribution
Sample sizes n=4, n=10
n = 4
n = 10
Slide 22


Central Limit Effect –
Histograms of sample means
Quadratic U Distribution
$Y = 12\left(x - \frac{1}{2}\right)^2, \; x \in [0,1]$
Area = 1
n = 1
n = 2
Sample means from sample size
n=1, n=2,
500 samples
Slide 23


Central Limit Effect -- Histograms of sample means
Quadratic U Distribution
Sample sizes n=4, n=10
n = 4
n = 10
Slide 24

Central Limit Theorem – heuristic formulation

Central Limit Theorem:
When sampling from almost any distribution,
$\bar{X}$ is approximately Normally distributed in large samples.

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Central Limit Theorem – theoretical formulation

Let $\{X_1, X_2, \ldots, X_k, \ldots\}$ be a sequence of independent observations from one specific random process. Let and $E(X) = \mu$ and $SD(X) = \sigma$ and both be finite ($0 < \sigma < \infty$; $|\mu| < \infty$). If $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$, sample-avg,

Then $\bar{X}$ has a distribution which approaches $N(\mu, \sigma^2/n)$, as $n \to \infty$.

The Distribution of a Linear Combination

Linear Combination

Given a collection of n random variables $X_1, \ldots, X_n$ and n numerical constants $a_1, \ldots, a_n$, the rv

$$Y = a_1 X_1 + \ldots + a_n X_n = \sum_{i=1}^{n} a_i X_i$$

is called a *linear combination* of the X_i's.

Expected Value of a Linear Combination

Let $X_1, \ldots, X_n$ have mean values $\mu_1, \mu_2, \ldots, \mu_n$ and variances of $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively

Whether or not the X_i's are independent,

$$E(a_1 X_1 + \ldots + a_n X_n) = a_1 E(X_1) + \ldots + a_n E(X_n)$$

$$= a_1 \mu_1 + \ldots + a_n \mu_n$$

Variance of a Linear Combination

If $X_1, \ldots, X_n$ are independent,

$$V(a_1 X_1 + \ldots + a_n X_n) = a_1^2 V(X_1) + \ldots + a_n^2 V(X_n)$$

$$= a_1^2 \sigma_1^2 + \ldots + a_n^2 \sigma_n^2$$

and

$$\sigma_{a_1 X_1 + \ldots + a_n X_n} = \sqrt{a_1^2 \sigma_1^2 + \ldots + a_n^2 \sigma_n^2}$$

Variance of a Linear Combination

For any $X_1,\ldots, X_n$,

$$V(a_1X_1+\ldots+a_nX_n)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_ia_j\text{Cov}(X_i,X_j)$$

Difference Between Two Random Variables

$$E(X_1-X_2)=E(X_1)-E(X_2)$$

and, if X_1 and X_2 are independent,

$$V(X_1-X_2)=V(X_1)+V(X_2)$$

Difference Between Normal Random Variables

If X_1, X_2,…X_n are independent, normally distributed rv's, then any linear combination of the X_i's also has a normal distribution. The difference $X_1 - X_2$ between two independent, normally distributed variables is itself normally distributed.