StatsM254 Statistical Methods in Computational Biology Lecture 11b - 05/21/2014

Lecture 11b

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Ising Model Example

A motivating example for the Gibbs sampler and the Metropolis-Hasting's algorithm is an Ising model on a circle. Consider m particles spaced around a circle. Each particle i can be in one of two states $X_i \in \{-1, 1\}$ and $X = (X_1, \dots, X_m)^T$.
Our target distribution is

$$\Pi(X) \propto e^{\beta \sum_{i=1}^{m-1} X_i X_{i+1}}$$

Approach 1: Metropolis Hastings Algorithm

Propose $q(X, X')$ as:

- Randomly pick a dimension in X
- Flip its sign and name it as X'

The acceptance probability will be

$$\begin{aligned}\alpha(X, X') &= min\{\frac{\Pi(X')q(X', X)}{\Pi(X)q(X, X')}, 1\} \\ &= min\{\frac{\exp(\beta \sum_{|j-i|\geq 2} X_j X_{j+1} - \beta(X_{i-1}X_i + X_i X_{i+1}))\frac{1}{m}}{\exp(\beta \sum_{|j-i|\geq 2} X_j X_{j+1} + \beta(X_{i-1}X_i + X_i X_{i+1}))\frac{1}{m}}, 1\} \\ &= min\{e^{-2\beta(X_{i-1}X_i + X_i X_{i+1})}, 1\}\end{aligned}$$

Approach 2: Gibbs Sampler

Iteratively sample X_i given X_{-i} by the following conditional distribution:

$$\begin{aligned}P(X_i = 1|X_{-i}) &= \frac{P(X_{i=1,X_{-i}})}{P(X_{i=1}, X_{-i}) + P(X_{i=-1}, X_{-i})} \\ &= \frac{\exp(\beta \sum_{|j-i|\geq 2} x_i x_{i+1} + \beta(x_{i-1} + x_{i+1}))}{\exp(\beta \sum_{|j-i|\geq 2} x_j x_{j+1} + \beta(x_{i-1} + x_{i+1})) + \exp(\beta \sum_{|j-i|\geq 2} x_j x_{j+1} - \beta(x_{i-1} + x_{i+1}))} \\ &= \frac{e^{\beta(x_{i-1}+x_{i+1})}}{e^{\beta(x_{i-1}+x_{i+1})} + e^{-\beta(x_{i-1}+x_{i+1})}}\end{aligned}$$