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SIMPLE RULES FOR OPTIMAL PORTFOLIO SELECTION:
THE MULTI GROUP CASE

*Edwin J. Elton, Martin J. Gruber, and Manfred W. Padberg**

The inception of modern portfolio theory dates from Markowitz's pioneering article [7] and subsequent book [8]. Yet despite the early development of a full theory of portfolio management, this theory has rarely been implemented. One problem arises from the difficulty in generating inputs to the general portfolio model. Index models and simple structures for correlation relationships, which go a long way towards solving this problem, have been developed. Yet the time and cost of solving actual portfolio problems (involving the solution of a quadratic programming problem) and more importantly the difficulty of educating portfolio managers to relate to risk return trade-offs in terms of covariances has virtually brought the application of portfolio theory to a halt.

In an earlier paper [3] we showed that if one is willing to accept the existence of a risk-free asset and is willing to either:

- 1) assume that the single index model adequately describes the variance-covariance structure; or
- 2) assume that a single number is a good estimate of all pair-wise correlation coefficients¹

then a simple decision rule (which does not involve an iterative algorithm) can be derived for the selection of optimal portfolios. Furthermore, this simple decision rule does not involve covariances or correlations and is formulated in terms to which the portfolio manager should be able to relate.

The purpose of this paper is to extend the development of simple decision rules to cases where more complex models are used to represent the correlation structure between stocks. Two cases will be examined. One is a multi-group model, which assumes that the correlation coefficients between any firm in one group and all other firms are identical for members of the same group. This was selected since we have shown in an earlier study that this technique

* *New York University.*

¹For evidence on the ability of a single number to represent correlation structures, see Elton and Gruber [4].

provided useful forecasts of future correlation structures. These forecasts were judged useful in two ways. They led to more accurate estimates of actual future correlation coefficients than the single index model, multiple index models, the use of an overall average correlation coefficient, and historic pairwise correlations. Second, they led to the selection of portfolios which proved more efficient (in future periods) than the above mentioned techniques. Hence decision rules for portfolio selection when the multi-group model is used will be examined in this paper.

The second case that will be explored is one in which a particular multi-index model is used to represent the correlation structure between securities. Multi-index models have gained attention because of their ability to account for more of the covariance structure than single index models. While there are many forms of multi-index models, the one we have chosen to explore is the diagonal form first presented in Cohen and Pogue [1].

This paper is divided into two sections according to the two models of estimating the covariance structure between securities described above.

I. Multiple Group Models

In [4] the authors presented a simplified structure for the correlation matrix which did an excellent job of forecasting future correlation matrices and led to the selection of efficient portfolios. The structure rested on the assumption that the correlation matrix could be partitioned into submatrices where all correlation coefficients within a submatrix are the same but the value of the correlation coefficient might differ between the submatrices. For example, if there were two industries in our sample--chemicals and steels--then this assumption implies that the correlation coefficient between all steels is the same constant (ρ_{ss}); that the correlation coefficient between all chemicals is the same (ρ_{cc}) but potentially different from the correlation coefficient for steels; and that the correlation coefficient between a steel firm and a chemical firm is still a third constant (ρ_{cs}).² This is illustrated in Figure 1. In this section, we will derive simple decision rules for the construction of optimal portfolios when the correlation structure between securities is described by this multi-group model. Once again we will separate the case where short selling is allowed from the case where short selling is forbidden.

²Grouping was performed in two ways: along traditional industry lines and by using a varimax rotation of factor loadings of historic rate of return data. See Elton and Gruber [4] for a more detailed discussion of the methodology and results.

	STEELS	CHEMICALS
STEELS	ρ_{SS}	ρ_{CS}
CHEMICALS	ρ_{CS}	ρ_{CC}

Figure 1

A. Short Selling Allowed

Let us define:

1. ρ_{kk} = the correlation coefficient between members of group k
2. ρ_{kt} = the correlation coefficient between members of group k and t
3. σ_i = the standard deviation of security i
4. σ_{ij} = the covariance between security i and security j
5. \bar{R}_i = the expected return on security i
6. R_f = the risk-free rate of interest
7. \bar{R}_p = the expected rate of return on the optimal portfolio
8. σ_p = the standard deviation of the optimal portfolio
9. N_k = the number of securities in group k
10. X_k = the set of stocks in group k
11. P = the number of groups
12. M_i = the fraction of funds invested in security i

If we allow short sales and assume the existence of a riskless asset, then the appropriate objective function is to maximize θ the excess return on the portfolio divided by the standard deviation of the portfolio.³ The first order conditions necessary for a maximum are presented by Lintner [6].⁴ They are

$$(1) \quad Z_i \sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^N Z_j \sigma_{ij} = \bar{R}_i - R_f \quad i = 1, \dots, N$$

where $Z_i = M_i (\bar{R}_p - R_f) / \sigma_p^2$.

³We are following Lintner's [6] suggestion in treating short sales. That is, the short seller pays any dividends which accrue to the person who lends him the stock and gets a capital gain (or loss) which is the negative of any price appreciation. In addition the short seller is assumed to receive interest at the riskless rate on both the money loaned to the owner of the borrowed stock and the money placed in escrow when the short sale is made. See Lintner [6] for a full discussion of these assumptions.

⁴See Lintner [6] for a proof that this is the correct objective function.

For a security i , which is a member of group k , equation (1) can be written as

$$(2) \quad Z_i \sigma_i^2 (1 - \rho_{kk}) + \sigma_i \sum_{g=1}^p \rho_{kg} \phi_g = \bar{R}_i - R_f$$

where we have set $\phi_g = \sum_{j \in X_g} \sigma_j Z_j$. Solving for Z_i yields

$$(3) \quad Z_i = \frac{1}{\sigma_i (1 - \rho_{kk})} \left[\frac{\bar{R}_i - R_f}{\sigma_i} - \sum_{g=1}^p \rho_{kg} \phi_g \right]$$

The above would be a solution if we can express the quantities ϕ_g in terms of the other variables in equation (3). This can be accomplished by multiplying each equation (3) (one of each value of g) by σ_i and summing over all members of a particular group. Rearranging the resulting expressions yields one equation for each group as follows:

$$(4) \quad (1 - \rho_{kk}) \phi_k + N_k \sum_{g=1}^p \rho_{kg} \phi_g = \sum_{j \in X_k} \frac{\bar{R}_j - R_f}{\sigma_j}$$

where the index k assumes all values $1, 2, \dots, p$. After dividing each equation (4) by the factor $(1 - \rho_{kk})$, this system of equations can be written in matrix

notation as $A \underset{\sim}{\phi} = C$ where A is a matrix of size $p \times p$ with elements

$$a_{kg} = \begin{cases} N_k \rho_{kg} / (1 - \rho_{kk}) & \text{if } k \neq g \\ 1 + N_k \rho_{kk} / (1 - \rho_{kk}) & \text{if } k = g \end{cases}$$

and where $\underset{\sim}{\phi}$ is the vector with p components ϕ_g $g=1, \dots, p$, and C is the vector with p components $\sum_{j \in X_g} (\bar{R}_j - R_f) / [\sigma_j (1 - \rho_{gg})]$ for $g=1, \dots, p$. The solution to the

system of equations (4) can thus be found by inverting the $p \times p$ matrix A . And using equation (3) we can thus determine the solution in terms of the variables Z_i .⁵ Note that equation (3) is of the form

⁵Note that no matter how many securities are considered the matrix to be inverted depends only on the number p of different groups considered. Since the number p will typically be small compared with the total number of securities involved, the computation is significantly simplified. Note also that the matrix need only be inverted once to solve for all cases involving a specific number of groups. For example, a specific solution to the matrix inversion can be performed for the four group case and this solution used on any problem involving four groups.

$$(5) \quad Z_i = \frac{1}{\sigma_i (1 - \rho_{kk})} \left[\frac{\bar{R}_i - R_f}{\sigma_i} - \psi_k \right]$$

where the constant ψ_k has the same value for all members of group k . The value of ψ_k is determined solely by the characteristics of the population of stocks under consideration. Since it does not depend on the composition of the optimal portfolio, its value can be computed before the analysis of the optimal portfolio is begun. Then it is a trivial task to calculate Z_i for all securities. In fact it can easily be done with pencil and paper in a few minutes.⁶ Note how easy it is to determine if any stock should be held long or sold short. If the excess return to standard deviation for any security is larger than its group constant, it should be bought; if it is smaller, it should be sold short.

The optimum amount to invest in each security M_i^o can be found quite easily by scaling the Z_i 's so that the sum of their absolute values adds to one

$$M_i^o = \frac{Z_i}{\sum_{j=1}^N |Z_j|}.$$

This completes our discussion of multiple group selection with short sales. It remains to examine the case of no short sales.

B. Short Sales Not Allowed

If short sales are not allowed, we have to make use of the Kuhn-Tucker conditions. In (3) we prove that they are both necessary and sufficient. The Kuhn-Tucker conditions for the problem of maximizing θ can be written as follows:

$$(6) \quad (\bar{R}_i - R_f) - Z_i \sigma_i^2 - \sum_{\substack{j=1 \\ j \neq i}}^N Z_j \sigma_{ij} + u_i = 0$$

$$(7) \quad Z_i \geq 0 \quad u_i \geq 0$$

$$(8) \quad Z_i u_i = 0$$

⁶It is interesting to note that the implementation of the multi-group case is no more difficult than the implementation of the single group case when short selling is allowed. See Elton, Gruber and Padberg [3] for a discussion of the single group case.

In this section we will use X_k' to denote the total set of securities in group K (formerly denoted by X_k). X_k will then refer to the subset of group X_k' which is in the optimal portfolio. As will become clear shortly, this change in notation allows us to use the equations of Section A for the case of no short sales. Solving for Z_i for members of group k yields

$$(9) \quad Z_i = \frac{\bar{R}_i - R_f}{\sigma_i^2 (1 - \rho_{kk})} - \frac{1}{\sigma_i} \sum_{g=1}^p \frac{\rho_{kg}}{1 - \rho_{kk}} \phi_g + \frac{u_i}{\sigma_i (1 - \rho_{kk})}$$

where we have again abbreviated $\phi_g = \sum_{j \in X_g} \sigma_j Z_j$ for $g=1, \dots, p$. First note

that $Z_i = 0$ for any security that is not in the optimal portfolio. Consequently, $\phi_g = \sum_{j \in X_g} \sigma_j Z_j$ for $g=1, \dots, p$, i.e., the summations can be taken over all

members of the subset X_g rather than over the set X_g' . Secondly, from the complementarity conditions (8), u_i is zero for all securities in the optimal portfolio. These two observations together imply that the system of equations (4) can again be used to solve for the quantities ϕ_g , for $g=1, \dots, p$. Substituting the solutions into equation (9) yields equation (5) with the additional term $\frac{u_i}{\sigma_i (1 - \rho_{kk})}$ or $Z_i = \frac{1}{\sigma_i (1 - \rho_{kk})} \left[\frac{\bar{R}_i - R_f}{\sigma_i} - \psi_k \right] + \frac{u_i}{\sigma_i (1 - \rho_{kk})}$.

The term containing the u_i can only increase Z_i . Hence if Z_i is positive with u_i equal to zero, a positive u_i cannot make $Z_i = 0$. Thus any security with positive Z_i when $u_i = 0$ must be included. Correspondingly, any security with negative Z_i when $u_i = 0$ must be excluded. It thus follows from the Kuhn-Tucker conditions, that within each group k the following properties hold: If a security with a particular value $(\bar{R}_i - R_f)/\sigma_i$ has a positive Z_i , then all securities with a higher excess-return-to-standard-deviation ratio also produce a positive Z_i . Similarly, if a stock produces a negative Z_i , all lower ranking stocks will also have a negative Z_i . Consequently, within each group of securities there is a group-specific security i such that all securities having a larger ratio $(\bar{R}_j - R_f)/\sigma_j$ will be in the optimal portfolio, whereas all securities having a lower ratio will not be in the optimal portfolio. In order to

determine an optimal portfolio, all that remains to be done is to find the cut-off rate in each group. The following seems to us to be an efficient method, since the number of different groups considered will typically be small.

1. Rank all stocks in each group by decreasing excess return to standard deviation. Go to 2.
2. Determine an optimal portfolio for stocks from group 1 only and mark group 1 as being checked. Compute the quantities Ψ_k for $k=1, \dots, p$. Go to 3.
3. If for the top ranking securities in the unchecked groups the term in the brackets in equation (5) is zero or negative or if there are no more unchecked groups, stop. Otherwise, pick any one of the unchecked groups, for which the expression (5) is positive, say group q , and go to 4.
4. Determine an optimal portfolio among the stocks from the groups already checked and the current group q in such a fashion that whenever a new stock from group q is included, optimality for the checked groups is reestablished immediately. (This can be accomplished by a modification of any "back-tracking" scheme.) Once an optimal solution for all checked groups is attained including group 1, mark group q as being checked and go to 3.

The procedure determines the cut-off rates and the values for Z_i . The optimal amount to invest in each security is determined by dividing each Z_i by the sum of the Z_i 's. The crucial step in the procedure is, of course, the re-optimization called for in step 4 whenever one considers a new stock in the current group q for inclusion in the "current" optimal portfolio. The back-tracking procedure must make use of the fact that all of the currently considered stocks in group q remain in the portfolio while adjustments (addition/deletions) are made only in groups already checked. In Appendix A we state formulas for adding/dropping securities in a particular group and we will illustrate the procedure next by means of a numerical example.

Example 1: Suppose that we have two different groups with eight and seven securities, respectively. The excess returns to standard deviation for each security in each group are given by the data in Table 1. Furthermore, let $\rho_{11} = 1/2$, $\rho_{12} = 1/3$, and $\rho_{22} = 2/5$. Starting with group 1, we find that securities 1, 2, and 3 constitute an optimal portfolio if one considers group 1 only. Using formula (A-1) of Appendix A, we find that $\Psi_1 = 6$ and $\Psi_2 = 4$.

TABLE 1								
Group 1	10	7	7	6	4	3	3	2
Group 2	8	4.5	4	4	3	2	1	X

Furthermore, the auxiliary quantities for the recalculation of the Ψ_q are given by (A-2) and we obtain $a_{11} = 1/8$, $a_{12} = 1/12$, and $a_{22} = 7/30$. We proceed now

with group 2 and find that security 1 of group 2 must be included since $8 - 4 > 0$. Computing the new Ψ_g by (A-1) we find $\Psi_1 = 6 \ 8/35$ and $\Psi_2 = 4 \ 16/25$. Thus the optimum portfolio consists of the first three securities in group 1 and the first security of group 2.

II. The Diagonal Form of the Multi-Index Model

Cohen and Pogue [1] have presented a multi-index model that leads to a diagonal form for the covariance structure between securities. The assumptions underlying the model are that each stock is linearly related to one group index and that each group index is linearly related to a market index. This model can be represented as

$$R_i = \alpha_i + \beta_i J_j + \epsilon_i$$

$$J_j = \gamma_j + b_j I_m + C_j$$

$$I_m = a + d$$

$$E(\epsilon_i \epsilon_k) = 0 \quad i=1, \dots, N \quad k=1, \dots, N \quad i \neq k$$

$$E(C_j C_\ell) = 0 \quad j=1, \dots, P \quad \ell=1, \dots, P \quad j \neq \ell$$

$$E(\epsilon_i C_j) = 0 \quad i=1, \dots, N \quad j=1, \dots, P$$

$$E(\epsilon_i d) = 0 \quad i=1, \dots, N$$

$$E(C_j d) = 0 \quad j=1, \dots, P$$

where

1. R_i = the return on security i which is in group j (a random variable)
2. J_j = the return on the index for group j
3. I_m = the market index
4. β_i = a measure of the responsiveness of security i to changes in the group index J_j
5. α_i = the return on security i that is independent of the group index
6. ϵ_i = a variable with mean of zero and variance $\sigma_{\epsilon_i}^2$ which measures the variance of security i not associated with changes in the group (or market) index
7. b_j = a measure of the responsiveness of index j to changes in the market index
8. γ_j = the return on index j that is independent of the market index
9. C_j = a variable with a mean of zero and a variance of $\sigma_{C_j}^2$ which measures the variance of group j not associated with changes in the market index

10. σ_m^2 = the variance of the market index
11. a = the mean return of the market
12. d = a variable with a mean of zero and variance equal to σ_m^2
13. P = the number of group (indices) which are appropriate

These equations make clear the approximations of the diagonal form of the multi-index model to the variance covariance structure. While each stock is linearly related to one group index and all group indices are linearly related to the market, the residuals from any of these relationships are assumed to be uncorrelated.

In this section we shall develop simple decision rules for portfolio composition when this diagonal form of the multi-index model is assumed to be a reasonable way to forecast future correlation coefficients. We shall separate the case where short selling is permitted from the case where it is not allowed.

A. Short Selling Allowed

If short selling is allowed, and the existence of a riskless asset is assumed, then the portfolio manager's task is to find the portfolio that has the largest excess return to risk. We can employ the general equation (1) originally presented by Lintner to solve this problem.

For a security i which is affected by group index k , equation (1) can be written as

$$(10) \quad Z_i \sigma_{\epsilon_i}^2 + \beta_i \sum_{g=1}^P d_{gk} \phi_g = \bar{R}_i - R_f$$

where

$$1. \quad d_{gk} = b_k b_g \sigma_m^2 \quad \text{for } g \neq k$$

$$2. \quad d_{kk} = b_k^2 \sigma_m^2 + \sigma_k^2 \quad \text{for } g = k$$

$$3. \quad \phi_g = \sum_{j \in X_g} \beta_j Z_j$$

Solving for Z_i yields

$$(11) \quad Z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - \sum_{g=1}^P d_{gk} \phi_g \right]$$

The above would be a solution if we can express the quantities ϕ_g in terms of known parameters. This can be accomplished by multiplying each equation (11)

by β_j and summing over all members of the group. Performing this summation and rearranging yields the following equation for each group:

$$(12) \quad \phi_k + \sum_{j \in X_k} \frac{\beta_j^2}{\sigma_{\epsilon_j}^2} \sum_{g=1}^p d_{gk} \phi_g = \sum_{j \in X_k} \frac{(\bar{R}_j - R_f) \beta_j}{\sigma_{\epsilon_j}^2}.$$

This system can be written in matrix notation $A \underset{\sim}{\phi} = C$ where A is a $p \times p$ matrix with elements

$$a_{kg} = \begin{cases} \sum_{j \in X_g} \frac{\beta_j^2}{\sigma_{\epsilon_j}^2} d_{gk} & \text{if } k \neq g \\ 1 + \sum_{j \in X_k} \frac{\beta_j^2}{\sigma_{\epsilon_j}^2} d_{kk} & \text{if } k = g \end{cases}$$

and where $\underset{\sim}{\phi}$ is the vector with p components ϕ_g $g=1, \dots, p$ and C is a vector of

p components $\sum_{j \in X_g} \frac{(\bar{R}_j - R_f) \beta_j}{\sigma_{\epsilon_j}^2}$ for $g = 1, \dots, p$. Once again, the solution to

this system is found by inverting the matrix.⁷

Note that equation (11) has the form

$$(13) \quad Z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - \psi_k \right]$$

where ψ_k has the same value for all members of group k . This produces a set of arguments about selection and which securities are sold long or short, analogous to those presented in Section I except that the conclusions and arguments are reversed when β is negative.

B. Short Sales Not Allowed

If short sales are not allowed, we have to make use of the Kuhn-Tucker conditions. The general form of the Kuhn-Tucker conditions necessary to maximize θ have already been presented in equations (6), (7), and (8). We shall now apply them to the diagonal form of the multiple index model when there are

⁷This can be done in general notation rather than being problem specific.

P groups. Once again a change in notation allows us to utilize the equations derived in the previous section. Let x'_k denote the securities in set k and let X_k denote the set of included securities in set k. This change in notation allows us to use the equations for the case of no short sales presented in Section IIA. Solving for Z_i for members of group k yields

$$(14) \quad Z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - \sum_{g=1}^p d_{gk} \phi_g \right] + \frac{u_i}{\sigma_{\epsilon_i}^2}$$

where the d_{gk} are defined as before and

$$\phi_g = \sum_{j \in X_g} \beta_j Z_j.$$

Note that $Z_i = 0$ for any portfolio which is not in the optimal portfolio. Hence,

$$\phi_g = \sum_{j \in X_g} \beta_j Z_j = \sum_{j \in X_g} \beta_j Z_j;$$

the summation can be taken over all members of the set X_g rather than over the set X'_g . Also note that from the complementary

conditions, $u_i = 0$ for all securities in the optimal portfolio. From these two conditions the system of equations (10) can again be used to solve for the quantities ϕ_g , $g=1, \dots, p$. Substituting the solutions into (14) yields equation

$$(13) \text{ with the additional term } \frac{u_i}{\sigma_{\epsilon_i}^2} \text{ or}$$

$$(15) \quad Z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - \psi_k \right] + \frac{u_i}{\sigma_{\epsilon_i}^2}.$$

The term containing u_i can only increase Z_i . As pointed out in Section I, since $u_i > 0$ when $Z_i = 0$ and $u_i = 0$ when $Z_i > 0$, any security with positive Z_i when $u_i = 0$ must be included and any security with negative Z_i when $u_i = 0$ must be excluded. Hence equation (1) describes the optimal portfolio in the following manner: any security which has a positive value of Z_i should be included in the optimal portfolio; any security which has a negative value of Z_i should be excluded.

It follows from the above discussion and from the discussion in Section IB that within each group of securities there is a specific cut-off point such

that

1. all securities with a positive β_i and a value of $\frac{\bar{R}_i - R_f}{\beta_i}$ greater than the cut-off point are included
2. all securities with a positive β_i and a value of $\frac{\bar{R}_i - R_f}{\beta_i}$ less than the cut-off point are excluded
3. all securities with a negative β_i and a value of $\frac{\bar{R}_i - R_f}{\beta_i}$ less than the cut-off point are included
4. all securities with a negative β_i and a value of $\frac{\bar{R}_i - R_f}{\beta_i}$ greater than the cut-off point are excluded.

To define an optimal portfolio all that remains is to find the cut-off point for each group. A procedure directly analogous to that outlined in Section IB can be used, remembering that it is necessary to check both positive and negative Beta stocks for inclusion or exclusion at all steps.

Once the search procedure has been completed and all securities with positive Z_i 's are found, the fraction of funds to place in each security is

$$M_i^0 = \frac{Z_i}{\sum_{j \in X_1, \dots, X_p} Z_j}.$$

Once again the problem of considering a new security for inclusion in the optimal portfolio becomes rather simple. The above procedure will lead to a set of cut-off points for $(\bar{R}_i - R_f)/\beta_i$ for both positive and negative β securities for each group. These cut-off points allow us to determine quickly the effect on the optimal portfolio of a new security. For example if a new security for group k with a positive β is considered, then

1. if its $(\bar{R}_i - R_f)/\beta_i$ is less than ψ_k , it can safely be discarded, or
2. if its $(\bar{R}_i - R_f)/\beta_i$ is more than ψ_k , then it must be included and the optimum recalculated.

Even if new calculations are needed, the amount of computation needed is very small.

III. Conclusion

In this paper we have developed decision rules that allow one to reach optimum solutions to portfolio problems without resorting to any complicated nonlinear programming algorithms. Furthermore, the characteristics of a stock that make it desirable are readily understood and calculated. Since the assumptions necessary to apply the simplified computational procedure discussed in this paper have been shown elsewhere [4] to be the preferred method of obtaining

inputs to portfolio problems, the procedure discussed in this paper should find extensive application in the future.

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Appendix A

In this appendix we state the formulas that are necessary to update the quantities ψ_k for $k=1, \dots, p$ of Section I when additions/deletions of securities from a particular group k of stocks are considered. To derive the formulas, the reader should note that one has to invert the matrix equation $A \underset{\sim}{\phi} = C$ of Section I. However, this can be done explicitly for the number of groups being considered or implicitly if the procedure discussed earlier is followed. If the procedure discussed earlier is followed, ψ_k is updated continuously. Specifically, let us suppose that we wish to determine the new quantities ψ_k of equation (5) when a subset $\Delta_k \subseteq X_k$ of securities in group k is added. Denote by ψ_k^{new} the resulting quantities for $k=1, \dots, p$ whereas ψ_k denote the "current" values [the initial conditions are $\psi_g = 0$ for $g=1, \dots, p$]. Then

$$\psi_g^{new} = \psi_g + a_{gk} (\psi_k^{new} - \psi_k) / a_{kk}$$

$$(A-1) \quad \psi_k^{new} = \{(1-\rho_{kk}) \psi_k + a_{kk} \sum_{h \in \Delta_k} (\bar{R}_h - R_f) / \sigma_h\} / [1-\rho_{kk} + n_k a_{kk}]$$

where $n_k = |\Delta_k|$ denotes the number of securities added in group k and the a_{gk} are defined recursively as follows: Initially, let $a_{gj} = \rho_{gj}$ for all $g=1, \dots, p$ and $j=1, \dots, p$. After the subset $\Delta_k \subseteq X_k$ of n_k securities in group k has been added, the new quantities a_{gj} , denoted by a_{gj}^{new} , are computed as follows:

$$(A-2) \quad a_{gj}^{new} = a_{gj} - [n_k a_{gk} a_{kj}] / [1-\rho_{kk} + n_k a_{kk}]$$

where indices g and j assume (independently) all values $1, 2, \dots, p$, whereas the index k , of course, designates the group in which we have added the securities. Note that the initial a_{gj} are symmetric since $\rho_{gj} = \rho_{jg}$ and that by the transformation (A-2) symmetry is preserved, which implies that only the upper diagonal part must be calculated.

Suppose next that we want to drop a subset $\Delta_k \subseteq X_k$ of securities from a portfolio. Again, let $n_k = |\Delta_k|$ denote the number of such securities. The new quantities ψ_g and a_{gj} are computed as follows:

$$\psi_g^{new} = \psi_g + a_{gk} (\psi_k^{new} - \psi_k) / a_{kk}.$$

$$(A-3) \quad \psi_k^{new} = ((1-\rho_{kk}) \psi_k - a_{kk} \sum_{h \in \Delta_k} (\bar{R}_h - R_f) / \sigma_h) / [1-\rho_{kk} - n_k a_{kk}]$$

and the new quantities a_{gj} , denoted a_{gj}^{new} , are computed as follows:

$$(A-4) \quad a_{gj}^{new} = a_{gj} + (n_k a_{gk} a_{kj}) / (1-\rho_{kk} - n_k a_{kk}).$$

In order to rederive formulas (A-1) through (A-4), the reader should note that the matrix A to be inverted when a subset of securities $\Delta_k \leq X_k$ is added or dropped changes as follows: $A^{new} = A + u - v$ here u is a column vector satisfying $u_i = 0$ for $i \neq k$, $i=1, \dots, p$ $u_k = \pm u_k / (1-\rho_{kk})$ and v is a row vector with elements $v_i = \rho_{ki}$ for $i=1, \dots, p$. The inverse of A^{new} can be calculated by the formula:

$$[A + u - v]^{-1} = A^{-1} - [1/(1+vA^{-1}u)] (A^{-1}u) (vA^{-1}).$$

Observing that in the matrix equation of section I $A\phi = C$ the right-hand vector C changes in its k^{th} component by $\pm \sum_{h \in \Delta_k} (\bar{R}_h - R_f) / \sigma_h$, all the formulas of this appendix follow.

Appendix B

In this appendix, we state the formulas that are necessary to update the quantities ψ_k for $k = 1, \dots, p$ in Section II, when additions/deletions of securities from a particular group k of stocks are considered. They are derived in the same fashion as in Appendix A and can be stated as follows: Initially, let $\psi_k = 0$ for $k = 1, \dots, p$ and let $a_{gk} = b_g b_k \sigma_m^2$ for $g \neq k$ and $a_{kk} = \sigma_k^2 + b_k^2 \sigma_m^2$ for $k = 1, \dots, p$. When a subset $\Delta_k \subseteq X_k'$ of securities in a particular group k is added, the new quantities ψ_g are obtained by

$$(B-1) \quad \psi_g^{new} = \psi_g + (a_{gk}/a_{kk}) (\psi_k^{new} - \psi_k)$$

$$\psi_k^{new} = \{ \psi_k + a_{kk} \sum_{h \in \Delta_k} \beta_h (\bar{R}_h - R_f) / \sigma_{\epsilon_h}^2 \} / [1 + \sum_{h \in \Delta_k} \frac{\beta_h^2}{\sigma_{\epsilon_h}^2} a_{kk}]$$

The auxiliary quantities a_{gj} are updated as follows:

$$(B-2) \quad a_{gj}^{new} = a_{gj} - [(\sum_{h \in \Delta_k} \frac{\beta_h^2}{\sigma_{\epsilon_h}^2}) a_{gk} a_{kj}] / [1 + (\sum_{h \in \Delta_k} \frac{\beta_h^2}{\sigma_{\epsilon_h}^2}) a_{kk}]$$

where the indices g and j assume all values $1, 2, \dots, p$. Note that again, due to symmetry, one needs to compute only the upper diagonal part of the coefficients. Similarly, if a subset of securities $\Delta_k \subseteq X_k'$ of group k is dropped, the quantities ψ_g must be recalculated as follows:

$$(B-3) \quad \psi_g^{new} = \psi_g + (a_{gk}/a_{kk}) (\psi_k^{new} - \psi_k)$$

$$\psi_k^{new} = \{ \psi_k - a_{kk} \sum_{h \in \Delta_k} \beta_h (\bar{R}_h - R_f) / \sigma_{\epsilon_h}^2 \} / [1 + \sum_{h \in \Delta_k} \frac{\beta_h^2}{\sigma_{\epsilon_h}^2} a_{kk}]$$

The auxiliary quantities a_{gj} are recalculated as follows:

$$(B-4) \quad a_{gj}^{new} = a_{gj} + [(\sum_{h \in \Delta_k} \frac{\beta_h^2}{\sigma_{\epsilon_h}^2}) a_{gk} a_{kj}] / [1 + \sum_{h \in \Delta_k} \frac{\beta_h^2}{\sigma_{\epsilon_h}^2} a_{kk}]$$