Simple Criteria for Optimal Portfolio Selection: Tracing Out the Efficient Frontier

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In a series of papers [1], [2], [3], and [4], we have shown that under alternative sets of assumptions about the form of the variance covariance structure of common stock returns, simple ranking devices can be used to determine optimal portfolios. These simple ranking devices have two advantages. First, the characteristics of a stock that make it desirable are unique to an individual stock and easily understood by portfolio managers. Secondly, the optimum portfolio is easy to determine and can normally be found with pencil and paper or at worst a hand calculator.

In each of these papers, we assumed the existence of a risk free asset and hence a unique optimum portfolio. This was not necessary. The purpose of this paper is to show how this assumption can be relaxed and our simple technique used to generate the full efficient frontier. In particular, we will show how the simple techniques described in the above papers can be used to find all corner portfolios. Since portfolios intermediate to corner portfolios are linear combinations of corner portfolios, this technique allows the construction of the full efficient frontier. In this paper, we will demonstrate how to find the efficient frontier for two cases: the single index model and a model assuming the correlation coefficient between all stocks is identical. We will examine both the case where short selling is allowed and the case where it is disallowed. The extension of the procedure described here to all of the other models described in [1], [2], [3] and [4] follows directly.

I. The Single Index Model and the Construction of Optimal Portfolios

In this section we shall assume that the standard single index model is an adequate description of reality. That is

1. $R_i = \alpha_i + \beta_i I + \epsilon_i$

2. $I = \alpha_{N+1} + \epsilon_{N+1}$

3. $E(\epsilon_{N+1}\epsilon_i) = 0 \qquad i = 1,\ldots,N$

4. $E(\epsilon_i\epsilon_j) = 0 \qquad i = 1,\ldots,N$
$$j = 1,\ldots,N$$
$$i \neq j$$

where R_i = the return on security i (a random variable)

I = a market index (a random variable)

β_i = a measure of the responsiveness of security i to changes in the market index

α_i = the return on security i that is independent of changes in the market index

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ϵ_i = a variable with a mean of zero and variance $\sigma_{\epsilon_i}^2$

σ_m^2 = the variance of the market index

The last two equations characterize the approximation of the standard single index model to the covariance structure. The assumption implied by these equations is that the only joint movement between securitites comes about because of a common response to a market index.

A. Short Sales Allowed

In (1), we showed that under the assumption of the single index model, one can solve the optimal portfolios with simple decision criteria without resorting to mathematical programming. In particular, we showed that if short sales were allowed, the optimal amount to invest in each security (X_i^0) is[1]

$$X_i^0 = \frac{Z_i}{\sum_{j=1}^{N} |Z_i|} \tag{1}$$

where Z_i is defined by

$$Z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\overline{R}_i - R_f}{\beta_i} - \phi_N \right] \tag{2}$$

and ϕ_N is defined by

$$\phi_N = \sigma_m^2 \frac{\sum_{j=1}^{N} \left[\frac{\overline{R}_j - R_f}{\sigma_{\epsilon_j}^2} \beta_j \right]}{1 + \sigma_m^2 \sum_{j=1}^{N} \frac{\beta_j^2}{\sigma_{\epsilon_j}^2}} \tag{3}$$

These equations allow one to solve for the composition of the optimal portfolio given the ability of the investor to borrow and lend unlimited amounts of funds at that riskless rate. If such borrowing and lending is disallowed, the concept of a riskless rate can still stand as an artifact to trace out the entire efficient frontier. The efficient frontier can be traced out by solving (1), (2), and (3) for a sequence of arbitrarily selected R_f. For example, R_f could start at zero and vary by .1 to maximum $\overline{R}_i$.[2]

If instead one wished to trace out corner portfolios a different procedure is needed. To understand the procedure first note that the sign of Z_i depends on the

1. X_i can be negative for short sales. We are following Lintner's [6] suggestion in treating short sales. That is the short seller pays any dividends which accrue to the person who lends him the stock and gets a capital gain (or loss) which is the negative of any price appreciation. In addition the short seller is assumed to received interest at the riskless rate on both the money loaned to the owner of the borrowed stock and the money placed in escrow when the short sale is made. See Lintner [6] for a full discussion of these assumptions. Readers who believe that $\sum_{i=1}^{N} x_i = 1$ in the case of short sales can make this substitution in equation (1) and the solution still holds.

2. Throughout the analysis of the efficient frontier one must deal with the problem of establishing the lowest point on the frontier which is of interest. Since most analysis is performed in nominal terms and

term in the brackets in equation (2). Note also that the term ϕ_N in the brackets of equation (2) is a constant independent of i. Hence for positive β_i if $(\bar{R}_i - R_f)/\beta_i$ is greater than the constant the security is purchased long; if it is less, it is sold short; and if it is equal, it is neither purchased nor sold short.[3] It is this latter condition which determines a corner portfolio. For any security, this can occur only once. This holds because $(\bar{R}_i - R_f)/\beta_i$ is a linear function of R_f. Likewise, the second term in the brackets is also linear in R_f. Two linear functions will have at most one intersection in the range of interest.

Therefore, to determine the corner portfolios:

1. For each security i, solve equation (2) for the value of R_f which makes Z_i and hence X_i^0 zero. Each distinct value of R_f greater than the lower bound will be uniquely associated with one corner portfolio.
2. For each value of R_f defined in Step 1, we use equations (1), (2), and (3) to determine the value of all X_i^0 in that corner portfolio.
3. Solve for the return and risk for each corner portfolio determined in Step 2.

Portfolios with $\bar{R}_p$ between corner portfolios can be determined in the usual way.[4]

B. *Optimal Portfolios when Short Sales are Not Allowed*

When short sales are not allowed, Kuhn-Tucker conditions are required to determine an optimum. In (1), we showed that if unlimited lending and borrowing can take place at the rate R_f, the optimum amount to invest in security i is

$$X_i^0 = \frac{Z_i}{\sum_{j \in k} Z_j} \qquad \text{for all } Z_i > 0 \tag{4}$$

where

(A) k is the set of securities in the optimal portfolio.

(B) $$Z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - \phi_k \right] \tag{5}$$

(C) $$\phi_k = \sigma_m^2 \frac{\sum_{j \in k} \frac{\bar{R}_j - R_f}{\sigma_{\epsilon_j}^2} \beta_j}{1 + \sigma_m^2 \sum_{j \in k} \frac{\beta_j^2}{\sigma_{\epsilon_j}^2}} \tag{6}$$

since the investor is always free to hold cash the lowest point on the efficient frontier of interest will usually be that associated with a lending rate of zero. However nothing in the analysis restricts the user from defining lower regions on the efficient (stock) frontier by considering values of R_f below zero. In fact in the limit R_f can be examined as it approaches infinity.

3. The opposite set of statements can be made for negative β.

4. One desires corner portfolios since intermediate portfolios can be determined by calculating portfolios consisting of adjacent corner portfolios. This is also true when the R_f's are arbitrarily selected if adjacent portfolios differ by at most one security. Readers who believe $\sum_{i=1}^{N} x_i = 1$ in the case of short sales can trace out the full efficient frontier by determining and two arbitrary portfolios.

Before addressing ourselves to the problem of tracing the entire efficient frontier, it is useful to review the ranking procedure that permits one to determine an optimal portfolio for a particular riskless rate R_f. Suppose first that all β_j are positive. Then all we have to do to arrive at an optimal portfolio is to rank from 1 to N all securities by $(\bar{R}_i - R_f)/\beta_i$ (highest to lowest). Then compute a value for equation (6) as if the set k only contained the first security. Next we calculate Z_i from equation (5) setting $i = 2$. If $Z_2 \leqslant 0$, we stop. The optimal portfolio contains only the first security. Otherwise we compute ϕ_k from (6) letting the set k contain the first two securities. We proceed for $i = 3, 4, \ldots$ until Z_i computed from equation (5) turns negative. If it turns negative for the $j + 1^{st}$ security then the set k contains the first j securities. Since the set k arrived at this way satisfied the Kuhn-Tucker conditions, we have found a simple and fast way to define all securities in the set k. Once these securities are found the X_i^0 value for all securities in the set can be found simply by calculating the Z_i for each security from equation (5), dividing each Z_i by the sum of the Z_i over the set k. Note that from equation (5) the desirability of any security and the amount we invest in it will be uniquely related to its excess return to β ratio. Negative or zero betas can easily be handled by the methods discussed in (1).

If lending and borrowing at a riskless rate is disallowed, the general shape of the efficient frontier can be traced out by repeating the above analysis for alternative values of R_f. However, the task is simplified by finding all "corner" portfolios along the efficient frontier.

In order to determine critical values of the riskless rate R_f which determine a different set k of securities to be included in an optimal portfolio, we write $R_f = R_f^0 + \lambda$, where R_f^0 is a riskless rate for which we have previously determined an optimal portfolio. We are then interested in R_f for values of $\lambda \geqslant 0$. We observe that using (6), relation (5) can be rewritten for Z_i as follows:

$$Z_i(\lambda) = \alpha_i - \Psi_i \lambda \qquad (7)$$

α_i is given by $\alpha_i = Z_i(0) - \mu_i$

where μ is the Kuhn-Tucker multiplier which has a value equal to 0 when Z_i from equation (5) is zero and a value of $-Z_i$ when Z_i calculated from equation (5) is negative.

$$\Psi_i \text{ is given by } \Psi_i = \frac{1}{\sigma_{\epsilon_i}^2} - \beta_i \frac{\sigma_m^2}{\sigma_{\epsilon_i}^2} \left[\frac{\displaystyle\sum_{j \in k} \frac{\beta_j}{\sigma_{\epsilon_j}^2}}{1 + \sigma_m^2 \displaystyle\sum_{j \in k} \beta_j^2 / \sigma_{\epsilon_j}^2} \right] \qquad (8)$$

In particular, $Z_i(\lambda)$ is a linear function of λ. Consequently, if for $R_f = R_f^0$ the optimal portfolio contains security i at a positive level Z_i, then security i will remain in an optimal portfolio with $R_f = R_f^0 + \lambda$ for a sufficiently small positive change λ provided that $\Psi_i \leqslant 0$. If, however, $\Psi_i > 0$, then security i is a candidate for leaving the optimal portfolio when a small positive change in λ is considered.

More precisely, letting K^+ denote the set of securities that are in the optimal portfolio at strictly positive level, we are interested in security j having $(\alpha_j/\Psi_j) = \min\{(\alpha_i/\Psi_i) \mid \Psi_i > 0, \ i \in K^+\}$, since it is the first candidate to leave the optimal portfolio. Similarly, letting K^- denote the set of securities having $\mu_i > 0$, i.e. for which $Z_i = (\beta_i/\sigma_{\epsilon_i}^2) \ [(\bar{R}_i - R_f)/\beta_i - \phi_k] < 0$ the first security to enter the optimal portfolio is given by the security e for which $(\alpha_e/\Psi_e) = \min\{(\alpha_i/\Psi_i) \mid \Psi_i < 0, \ i \in K^-\}$. Consequently, if $\Psi_j > \Psi_e$, security e will enter the optimal portfolio prior to security j leaving the optimal portfolio, whereas if $\Psi_j < \Psi_e$ security j will leave the optimal portfolio before e is considered for inclusion. So far we have not yet discussed securities for which $Z_i = 0$, which can be included in the optimal portfolio at zero level. Let us call the set of these securities K^0. Obviously, if Ψ_i as defined by (8) are all positive or zero for securities in K^0, one will not include any one of these securities in the optimal portfolio. However if $\Psi_i < 0$, some of these securities should enter the optimal portfolio. Prior to determining the next corner portfolio, one has to establish which securities in K^0 should enter the optimal portfolio at zero level. Once this is done, we can then determine the next critical value $\lambda' = \min \{(\alpha_j/\Psi_j),(\alpha_e/\Psi_e)\}$ where now Ψ_j and Ψ_e are calculated with respect to the possibly enlarged optimal portfolio.

The preceding readily defines a procedure for tracing out the entire efficent frontier. Starting with minimum R_f determine the optimal portfolio by the ranking procedure outlined above.[5] Classify all securities as "belonging" to the optimal portfolio or "not belonging" to the optimal portfolio including those for which formula (5) results in $Z_i = 0$. (In Appendix A we describe a procedure for carrying out this classification). Determine λ' and repeat.

Finally, the proportions in the optimal portfolio are calculated at every critical value of λ in the usual way, whereas optimal portfolios for "in between" values of λ are obtained by simple linear interpolation, i.e. as convex combinations of the respective corner portfolios.

II. Constant Correlation Coefficients and the Construction of Optimal Portfolios

In this section, we will assume that all pairwise correlation coefficients are equal. While this probably does not represent the true pattern one finds in the economy it is very difficult to obtain a better estimate. Elsewhere [5] we have shown that this assumption produces better estimates of future correlation coefficients than do historical correlation coefficients or those produced from the single index approximation discussed in section one. In fact the assumption of a constant correlation coefficient produced forecasts which were about as accurate as any of nine techniques tried in (5).

The derivation of the equations representing the optimal portfolio to hold with riskless borrowing and lending is given in (1). If short sales are allowed, the

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5. Zero is normally the lowest lending rate of interest since cash can be held. The analysis does not assume that funds can be borrowed at this rate since the efficient portfolio at all higher rates will be traced out. If the analysis is being conducted in real terms an appropriate lower level must be selected.

procedures outlined in Section IA of this paper are optimum if Z_i is defined as:

$$Z_i = \frac{1}{(1-\rho)\sigma_i} \left[\frac{\bar{R}_i - R_f}{\sigma_i} - \phi'_N \right]$$

where $\rho=$ the correlation coefficient between the returns on any two stocks
$\sigma_i=$ the standard deviation of the return on stock i

$$\phi'_N = \frac{\rho}{1-\rho+N\rho} \sum_{j=1}^{N} \frac{\bar{R}_j - R_f}{\sigma_j}$$

If short sales are disallowed, the procedures outlined in Section IB of this paper are optimum if Z_i is given by

$$Z_i = \frac{1}{(1-\rho)\sigma_i} \left[\frac{\bar{R}_i - R_f}{\sigma_i} - \phi'_k \right]$$

where

$$\phi'_k = \frac{\rho}{1-\rho+N\rho} \sum_{j \in k} \frac{\bar{R}_j - R_f}{\sigma_j}$$

and Ψ_i is given by

$$\Psi_i = \frac{1}{\sigma_i(1-\rho)} \left[\frac{1}{\sigma_i} - \frac{\rho}{1-\rho+k\rho} \sum_{j \in k} \frac{1}{\sigma_j} \right]$$

With these changes, the procedures follow directly from those outlined earlier.

III. CONCLUSION

In this paper, we have shown how the simple procedures constructed, under the assumption of unlimited lending and borrowing at the riskless rate, can be extended to trace out the full efficient frontier.

APPENDIX

Consider the set K^0 for which $Z_i = 0$ in equation (5). It is easy to show that these can be included or excluded at the R_f being considered since they do not effect ϕ_k. For any security in set K^0 it must be true that $(\bar{R}_i - R_f)/\beta_i$ is the same constant. Consider an increase in R_f by an amount δ. Then these can be ranked by $(\bar{R}_i - R_f - \delta)/\beta_i = (\bar{R}_i - R_f)/\beta_i - (\delta/\beta_i)$. Since $(\bar{R}_i - R_f)/\beta_i$ is the same they can be ranked by $(-\delta/\beta_i)$.

It follows that positive beta securities should be ranked in descending order of β_i and negative beta stocks in ascending order. Any remaining ties must be identical securities and the one(s) with the largest σ_q^2 can be excluded.

If $\Psi_i < 0$ for a security, it should be in the included set since an increase in λ would cause it to have a positive Z_i. Thus securities in set K^0 should be added to the included set k in decreasing order of β_i for positive Beta and ascending order for negative Beta until $\Psi_i \geq 0$ for the next security to be added. Furthermore an examination of Ψ_i shows that only either positive Beta securities or negative Beta securities would be added depending on the sign of $\sum_{i=1}^{k}(\beta_i/\sigma_{e_i}^2)$.

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