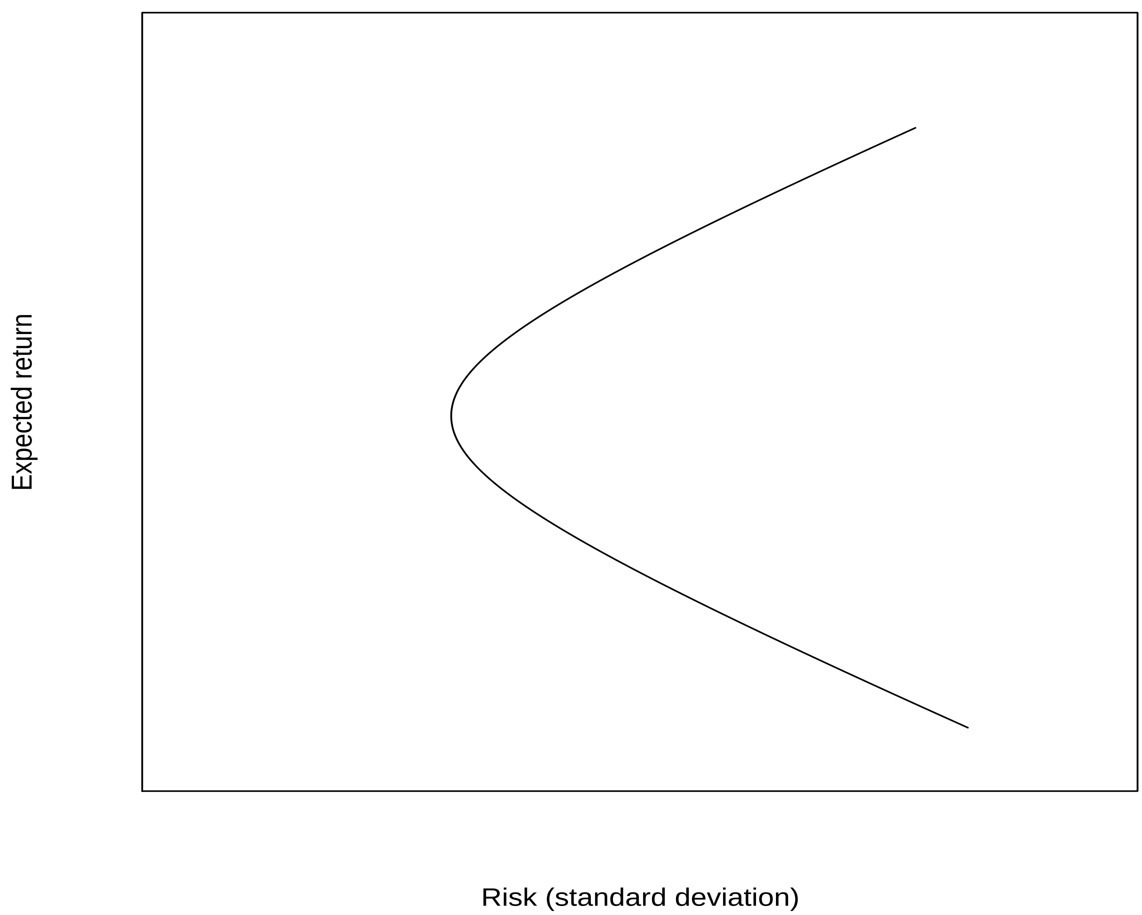
University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

The efficient frontier with short sales allowed and risk free lending and borrowing

Suppose riskless lending and borrowing exists. Let R_f be the return of the riskless asset (savings account, treasury bills, government bonds, etc.). We will examine the geometric pattern of combinations of the riskless asset and a risky portfolio.

Consider the portfolio possibilities curve below constructed using different combinations of two stocks. The efficient frontier is the concave part of the curve that begins with the minimum risk portfolio and extends to the maximum expected return portfolio.



Suppose now that the investor wants to invest a portion of her wealth on portfolio A (a point on the efficient frontier) and her remaining wealth on the riskless asset. Let x be the portion invested in portfolio A and $1-x$ the portion invested in the riskeless asset. This combination is a new portfolio. It has the following expected return and standard deviation:

$$\bar{R}_p = x\bar{R}_A + (1-x)R_f \tag{1}$$

and

$$\sigma_p = \sqrt{x^2\sigma_A^2 + (1-x)^2\sigma_f^2 + 2x(1-x)\sigma_{Af}}$$

However, because $\sigma_f = 0$ and $\sigma_{Af} = 0$ the standard deviation is simply:

$$\sigma_p = x\sigma_A \;\Rightarrow x = \frac{\sigma_p}{\sigma_A} \tag{2}$$

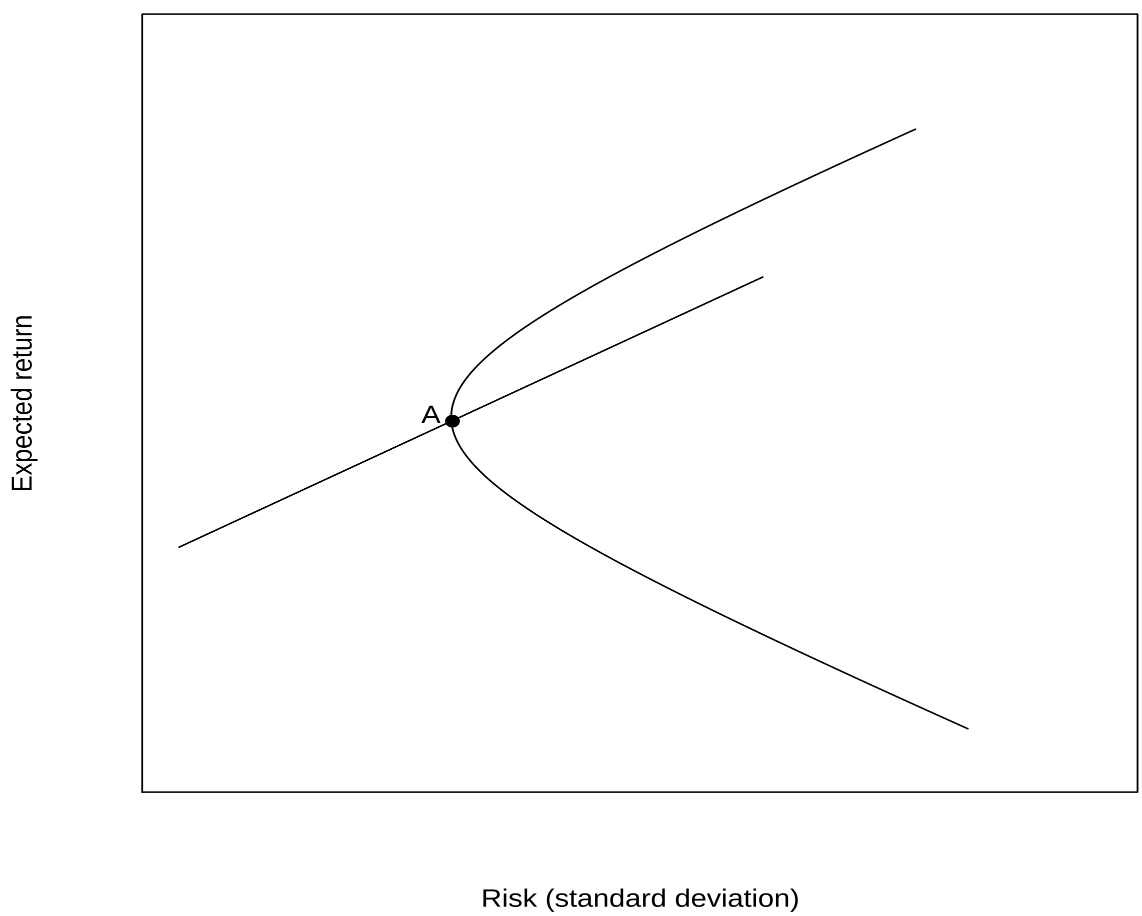
We substitute (2) into (1) to get:

$$\bar{R}_p = \frac{\sigma_p}{\sigma_A}\bar{R}_A + (1 - \frac{\sigma_p}{\sigma_A})R_f$$

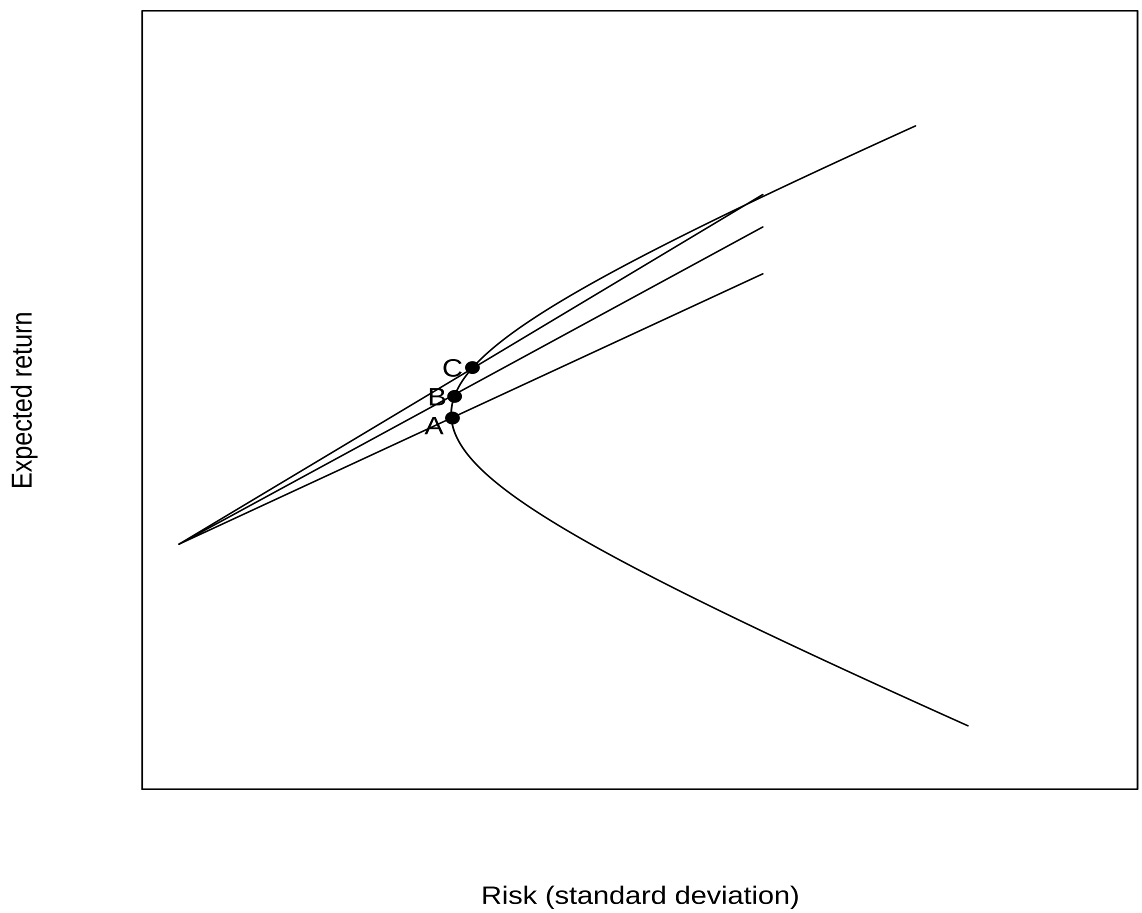
Finally,

$$\bar{R}_p = R_f + \left(\frac{\bar{R}_A - R_f}{\sigma_A}\right)\sigma_p \tag{3}$$

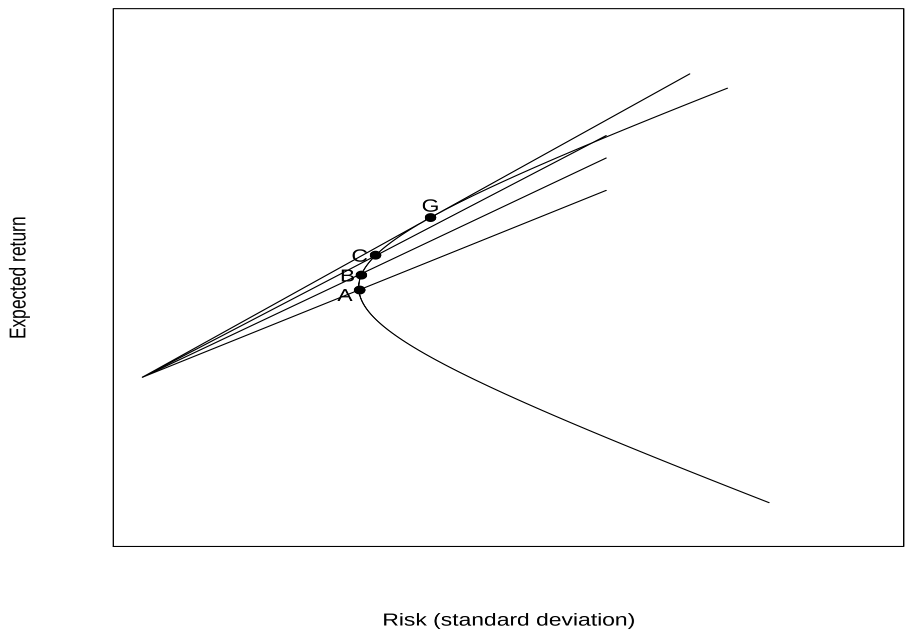
Equation (3) is a straight line that has intercept R_f, slope $\frac{\bar{R}_A - R_f}{\sigma_A}$, and it passes through portfolio A. See figure below.



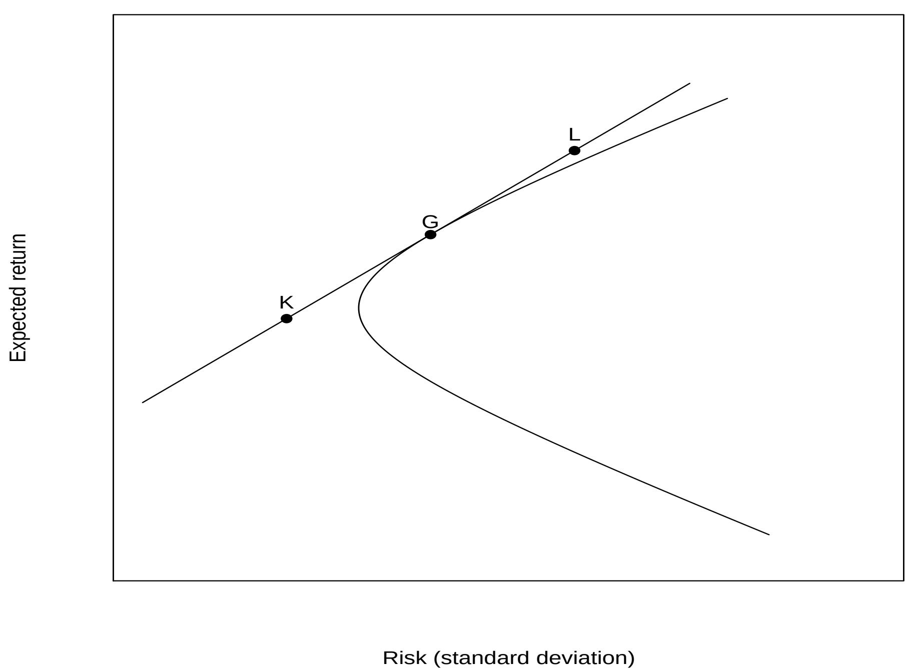
Portfolio A was an arbitrary point chosen on the efficient frontier. Another investor could choose portfolio B, or portfolio C. Which one of these combinations is best for the investor? Clearly, the combination between R_f and portfolio C. See figure below.



Therefore, the solution to this problem is to find the tangent of the line on the efficient frontier. The point of tangency (point G) is the solution.

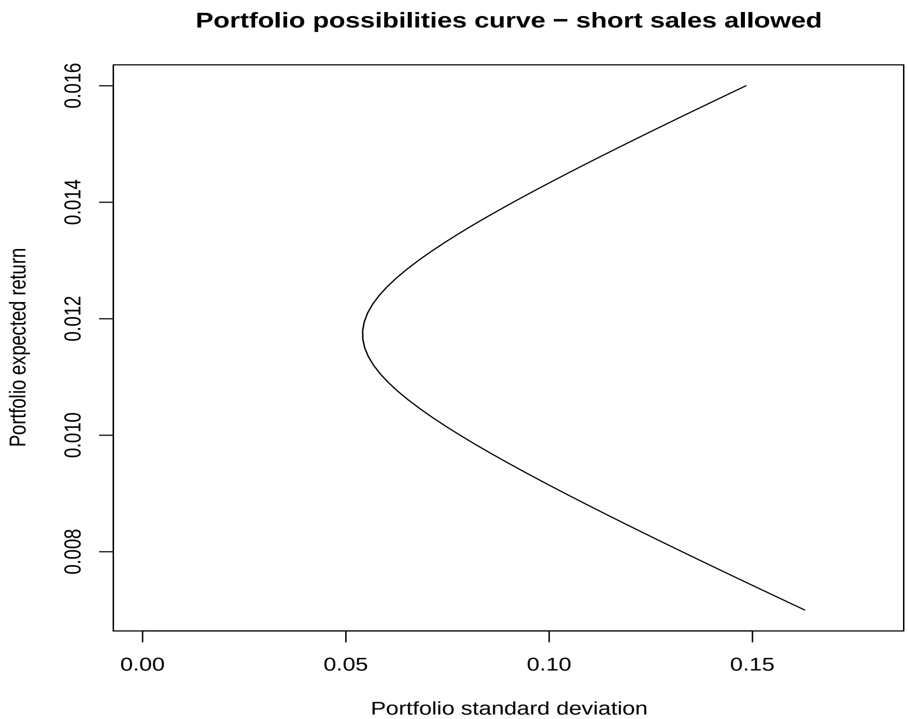


The investor now has the following choices: Invest all her wealth in portfolio G, or invest some of her wealth in portfolio G and the remaining in R_f (point K - lending), or borrow more funds to invest in portfolio G (point L). See figure below:



Example:

Suppose two stocks have the following expected return and variance: $\bar{R}_A = 0.01, \bar{R}_B = 0.013, \sigma_A^2 = 0.061, \sigma_B^2 = 0.0046$, and $\sigma_{AB} = 0.00062$. The portfolio possibilities curve is shown below:



Assume that the return of the riskless asset is $R_f = 0.008$. Approximate the composition of the point of tangency.