Quantum Field Theory as Generative Models
Emergent Gravity as Latent Variable Model

Tutorials for Statisticians and Machine Learning Researchers

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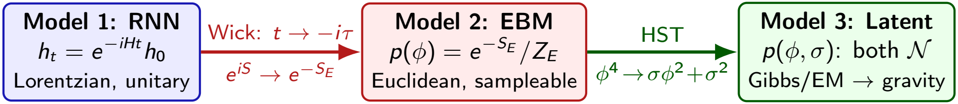
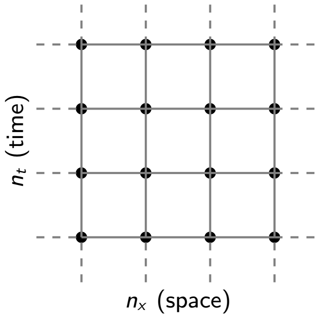
Three generative models



Model 1 (RNN): h_t is an extremely high-dimensional vector — a superposition of *all* basis vectors corresponding to all field configurations. h_t is exhaustive. A classical computer cannot simulate Model 1 efficiently.
Model 2 (EBM): we translate to Model 2 (via Wick rotation) for both analytical analysis and Monte Carlo sampling. This is where lattice QFT lives.
Model 3 (Latent): we introduce a latent variable to study emergence — especially in condensed matter physics and gravity.

The 4×4 lattice: 16 numbers on a grid



- $N_x = 4$, $N_t = 4$, $V = 16$ sites, periodic BC (torus)
- At each site: $\phi_{\mathbf{n}} \in \mathbb{R}$. Config: $\vec{\phi} \in \mathbb{R}^{\mathbf{16}}$
- Derivative $\to$ difference: $(\Delta_x \phi)_{\mathbf{n}} = \frac{\phi_{n_x+\mathbf{1}} - \phi_{n_x}}{a}$
- Difference operator $=$ 4×4 circulant matrix:

$$\Delta_x = \frac{1}{a}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

The Lagrangian: kinetic minus potential

$$\boxed{\mathcal{L} = \tfrac{1}{2}(\partial_t\phi)^2 - \tfrac{1}{2}(\partial_x\phi)^2 - \tfrac{1}{2}m^2\phi^2}$$

The $-$ sign between time and space = **Lorentzian** (Minkowski) signature.

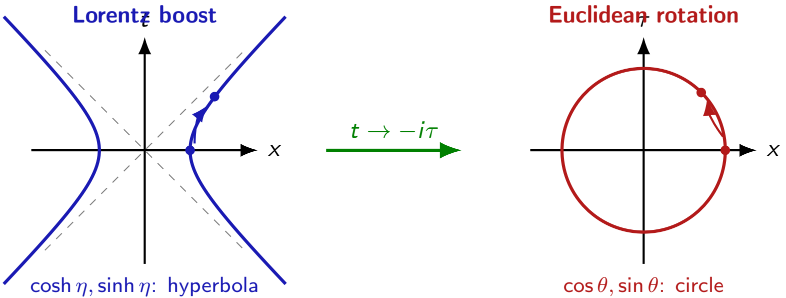
On the lattice: $S_M[\phi] = a^2\sum_{\mathrm{n}}\mathcal{L}_{\mathrm{n}} = \frac{1}{2}\vec{\phi}^{\,T}K_M\vec{\phi}$ (16 $\times$ 16 matrix).

In momentum space (FFT diagonalises K_M):

$$\boxed{(K_M)_{\mathrm{k}} = \hat{\omega}^2(k_t) - \hat{k}^2(k_x) - m^2} \quad \hat{\omega} = \frac{2}{a}\sin\frac{k_t a}{2}, \quad \hat{k} = \frac{2}{a}\sin\frac{k_x a}{2}$$

Indefinite: some eigenvalues $+$, some $-$. This is the lattice signature.

Lorentz boost vs Euclidean rotation



$(\partial_t\phi)^2 - (\partial_x\phi)^2 = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$: Lorentz scalar. Mass shell: $\omega^2 - k^2 = m^2$.
On lattice: Lorentz invariance is **emergent**, accurate to $O(a^2)$.

The path integral: transition amplitude

Transition amplitude from $\vec{\phi}_0$ at $t = 0$ to $\vec{\phi}_T$ at $t = T$:

$$\boxed{\langle \vec{\phi}_T | e^{-iHT} | \vec{\phi}_0 \rangle = \int \prod_{t=1}^{T-1} d\vec{\phi}_t \; e^{iS_M[\phi_{\mathbf{0}}, \phi_{\mathbf{1}}, \ldots, \phi_T]}}$$

Fix 4 values at $t=0$, fix 4 at $t=T$, sum over 8 intermediate.

In the quadratic form:

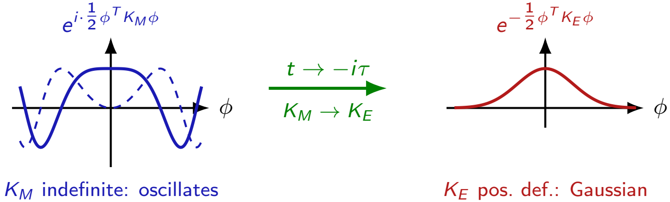
$$S_M[\phi] = \tfrac{1}{2} \vec{\phi}^T K_M \vec{\phi}, \qquad (K_M)_{\mathrm{k}} = \hat{\omega}^2 - \hat{k}^2 - m^2 \;\; (\textbf{indefinite})$$

$|e^{iS}| = |e^{i\frac{1}{2}\vec{\phi}^T K_M \vec{\phi}}| = 1$: oscillates, no probability.
Propagator: $\tilde{G}_M(\mathrm{k}) = 1/(K_M)_{\mathrm{k}}$, but $(K_M)_{\mathrm{k}}$ can be zero (on shell). Needs $i\epsilon$ to invert.

Wick rotation: $\frac{1}{2}\vec{\phi}^T K_M \vec{\phi} \;\to\; \frac{1}{2}\vec{\phi}^T K_E \vec{\phi}$

$t \to -i\tau$: $(\partial_t \phi)^2 \to -(\partial_\tau \phi)^2$. The minus between time and space disappears:

$$\boxed{S_E[\phi] = \tfrac{1}{2}\vec{\phi}^T K_E \vec{\phi}, \qquad (K_E)_{\mathrm{k}} = \hat{\omega}^2 + \hat{k}^2 + m^2 \;\; \textbf{(positive definite)}}$$



The path integrals:

$$Z_M = \int d^{16}\phi \; e^{i\frac{1}{2}\vec{\phi}^T K_M \vec{\phi}} \xrightarrow{t \to -i\tau} Z_E = \int d^{16}\phi \; e^{-\frac{1}{2}\vec{\phi}^T K_E \vec{\phi}}$$

$e^{-\frac{1}{2}\vec{\phi}^T K_E \vec{\phi}} \geq 0$ and integrable $\to$ defines a probability: $p(\phi) \propto e^{-S_E[\phi]} = e^{-\frac{1}{2}\vec{\phi}^T K_E \vec{\phi}}$ (energy-based model).

The side-by-side comparison on the 4×4 lattice

	Lorentzian (S_M)	**Euclidean (S_E)**
Sign pattern	$(\partial_t\phi)^2 - (\partial_x\phi)^2 - m^2\phi^2$	$(\partial_\tau\phi)^2 + (\partial_x\phi)^2 + m^2\phi^2$
Momentum	$(K_M)_{\mathbf{k}} = \hat{\omega}^2 - \hat{k}^2 - m^2$	$(K_E)_{\mathbf{k}} = \hat{\omega}^2 + \hat{k}^2 + m^2$
Eigenvalues	Some $+$, some $-$ (indefinite)	All > 0 (positive definite)
Each mode	$e^{i(K_M)_{\mathbf{k}}\|\hat{\phi}\|^2/2}$: phase	$e^{-(K_E)_{\mathbf{k}}\|\hat{\phi}\|^2/2}$: bell curve
Product	16 phases: not probability	16 Gaussians: $p(\phi) = \mathcal{N}(0, K_E^{-1})$
Symmetry	Lorentz: $\hat{\omega}^2 - \hat{k}^2$	Euclidean rotation: $\hat{\omega}^2 + \hat{k}^2$

Where Wick rotation acts:

$$\underbrace{\partial_t^2 - \partial_x^2}_{\text{wave (indefinite)}} \xrightarrow{t \to -i\tau} \underbrace{\partial_\tau^2 + \partial_x^2}_{\text{Laplacian (pos. def.)}}$$

Model 2: the energy-based model

$$\boxed{p(\phi) = \frac{e^{-S_E}}{Z_E}}$$

Energy-based model (EBM). Exponential family with natural parameters $m^2, 1/a^2$.

Model 1 (RNN) vs **Model 2 (EBM)**:

	Model 1: Lorentzian (RNN)	Model 2: Euclidean (EBM)
State	$h_t = e^{-iHt}\mathbf{h_0}$ (Fock vector)	$p(\phi) \propto e^{-S_E}$ (distribution)
Weight	e^{iS} (complex phase)	e^{-S_E} (real positive)
Sampling	Impossible (exhaustive sum)	MCMC / FFT-Gaussian

The free field: from e^{-S_E} to a 16-dimensional Gaussian

Start from the EBM:

$$p(\phi) \propto e^{-S_E[\phi]} = e^{-\frac{1}{2}\vec{\phi}^T K_E \vec{\phi}}, \qquad K_E = -\Delta_{\mathrm{lat}} + m^2 \ \ (16 \times 16,\ \text{pos. def.})$$

This is the density of a **16-dimensional Gaussian**:

$$\boxed{p(\phi) = \mathcal{N}(\vec{0},\ K_E^{-1}) = \frac{\sqrt{\det K_E}}{(2\pi)^8}\ e^{-\frac{1}{2}\vec{\phi}^T K_E \vec{\phi}}}$$

In momentum space: $p(\hat{\phi}) = \prod_{\mathsf{k}} \mathcal{N}(0, 1/(K_E)_{\mathsf{k}})$ (16 independent bell curves)

Covariance = propagator: $G_E = (K_E^{-1})_{\mathsf{nm}} = \langle \phi_{\mathsf{n}} \phi_{\mathsf{m}} \rangle = \mathrm{Cov}(\phi_{\mathsf{n}}, \phi_{\mathsf{m}})$
Log-normaliser (CGF): $\ln Z_E = 8 \ln(2\pi) - \frac{1}{2} \sum_{\mathsf{k}} \ln(K_E)_{\mathsf{k}}$ (Tr log!)
FFT sampling: $\eta_{\mathsf{k}} \sim \mathcal{N}(0,1)$, $\hat{\phi}_{\mathsf{k}} = \eta_{\mathsf{k}} / \sqrt{(K_E)_{\mathsf{k}}}$, inverse FFT. $O(V \log V)$.

The propagator/correlator: $G = K^{-1}$ on both sides

On both sides, the action is $S = \frac{1}{2}\vec{\phi}^T K \vec{\phi}$. The two-point function is K^{-1}:

Lorentzian: $G_M = K_M^{-1}$ (propagator = transition amplitude)

$$G_M(x, y) = \langle 0 | T\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle = \text{amplitude: particle } y \to x$$

$\tilde{G}_M(\mathsf{k}) = 1/(K_M)_{\mathsf{k}} = 1/(\hat{\omega}^2 - \hat{k}^2 - m^2 + i\epsilon)$ (**complex**; $i\epsilon$ needed since K_M has zero eigenvalues on shell)

Euclidean: $G_E = K_E^{-1}$ (correlator = covariance)

$$G_E(\mathsf{n}, \mathsf{m}) = \langle \phi_{\mathsf{n}} \phi_{\mathsf{m}} \rangle = \mathrm{Cov}(\phi_{\mathsf{n}}, \phi_{\mathsf{m}}) = (K_E^{-1})_{\mathsf{nm}}$$

$\tilde{G}_E(\mathsf{k}) = 1/(K_E)_{\mathsf{k}} = 1/(\hat{\omega}^2 + \hat{k}^2 + m^2)$ (**real**; K_E pos. def., no $i\epsilon$ needed)

Comparison: amplitude vs covariance

	Lorentzian: $G_M = K_M^{-1}$	**Euclidean:** $G_E = K_E^{-1}$
Action	$S_M = \frac{1}{2}\vec{\phi}^T K_M \vec{\phi}$	$S_E = \frac{1}{2}\vec{\phi}^T K_E \vec{\phi}$
K eigenvalues	$\hat{\omega}^2 - \hat{k}^2 - m^2$: $+$ and $-$	$\hat{\omega}^2 + \hat{k}^2 + m^2$: all > 0
K^{-1} exists?	Needs $i\epsilon$ (poles on shell)	**Yes**: routine 16×16 inverse
G values	Complex	Real, positive
Physical meaning	Transition amplitude	Covariance
Mass encoding	Pole (divergence at ω_k)	Decay rate ($e^{-m\tau}$)

The central formula on both sides: $G = K^{-1}$.
Why $G_E(\tau) \sim e^{-m\tau}$: at zero spatial momentum, Fourier-transform back to position: $G_E(\tau) = \int \frac{d\omega}{2\pi} \frac{e^{i\omega\tau}}{\omega^2 + m^2} = \frac{e^{-m|\tau|}}{2m}$ (residue at $\omega = im$). The mass $m =$ decay rate of the correlator. Fit the exponential $\to$ read off m.
Wick rotation: $K_M \to K_E$ (indefinite $\to$ positive definite), $G_M \to G_E$ (amplitude $\to$ covariance), pole $\to$ decay rate.
Same physics (m), different encoding. The Euclidean side is computable; the Lorentzian side is physical.

ϕ^4: non-Gaussian exponential family

$$\boxed{p(\phi) \propto e^{-S_E[\phi]} = e^{-\frac{1}{2}\vec{\phi}^T K_E \vec{\phi} - \frac{\lambda}{4!}\sum_{\mathbf{n}} \phi_{\mathbf{n}}^4}}$$

Non-Gaussian exponential family. **Natural parameters:** $\theta = (m^2, \lambda)$. **Sufficient statistics:** $(\phi_{\mathrm{n}}^2, \phi_{\mathrm{n}}^4)$.

Add source J (the "external field", like tilting in exp family):

$$p(\phi|J) \propto e^{-S_E[\phi] + \sum_{\mathbf{n}} J_{\mathbf{n}} \phi_{\mathbf{n}}} = e^{-\frac{1}{2}\phi^T K_E \phi - \frac{\lambda}{4!}\sum \phi^4 + J^T \phi}$$

Log-normaliser = cumulant generating function: $W[J] = \ln Z_E[J] = \ln \int d^{16}\phi \; e^{-S_E + J^T\phi}$
Mean parameters from $W[J]$:

$$\left.\frac{\partial W}{\partial J_{\mathrm{n}}}\right|_0 = \langle \phi_{\mathrm{n}} \rangle = 0, \quad \left.\frac{\partial^2 W}{\partial J_{\mathrm{n}} \partial J_{\mathrm{m}}}\right|_0 = \mathrm{Cov}(\phi_{\mathrm{n}}, \phi_{\mathrm{m}}) = G_E(\mathrm{n}, \mathrm{m})$$

First cumulant = mean. Second cumulant = covariance = **propagator**.
Feynman diagrams: Taylor-expand $e^{-\lambda\phi^4/(4!)}$; each term = Gaussian integral (Isserlis/Wick theorem).

Renormalization: the lattice-first view

On the 4×4 lattice: everything finite. No UV divergences.

Rendering stability:

$$\boxed{\mathcal{O}(\theta(a),\, a,\, \lambda) = \mathcal{O}_{\mathrm{phys}} + o(a/\lambda)}$$

Natural parameters $\theta(a) = (m^2(a), \lambda(a))$ run with a.
Mean parameters $\mathcal{O}_{\mathrm{phys}}$ (physical mass, scattering) don't.

Hubbard-Stratonovich Transformation (HST): decouple a quartic into two quadratics

ϕ^4: non-Gaussian

Both conditionals $\mathcal{N}$

Key identity: $e^{-g\phi^4/2} = \int \frac{d\sigma}{\sqrt{2\pi g}} e^{-\sigma^2/(2g)} e^{-\sigma\phi^2}$

	VAE	**RBM**	**HST**
Decoder $p(x\|z)$	Neural net	Bernoulli	$\mathcal{N}(0, K[\sigma]^{-1})$
Encoder $p(z\|x)$	**Intractable**	Bernoulli	$\mathcal{N}(-g\phi^2, g)$
Both tractable?	No	Yes	**Yes**

The T^2 interaction

(1) Stress-energy tensor for a scalar: $T_{\mu\nu}[\phi] = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}[(\partial\phi)^2 + m^2\phi^2]$

(2) Lagrangian: free + T^2 interaction: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\kappa^2}{2}T_{\mu\nu}T^{\mu\nu}$

(3) Model 2: $p(\phi) \propto e^{-\frac{1}{2}\phi^T K_0\phi - \frac{\kappa^2}{2}T^2}$ (non-Gaussian, quartic in ϕ, hard to sample)

(4) HST decoupling (exact identity, step by step):

$$e^{\frac{\kappa^2}{2}T_{\mu\nu}T^{\mu\nu}} = \int \frac{d\sigma}{\sqrt{2\pi\kappa^2}}\, e^{-\frac{\sigma_{\mu\nu}\sigma^{\mu\nu}}{2\kappa^2} + \sigma_{\mu\nu}T^{\mu\nu}[\phi]}$$

The coupling $\sigma_{\mu\nu}T^{\mu\nu}$ shifts Lagrangian of ϕ:

$$\begin{aligned} \tfrac{1}{2}\phi^T K_0\phi - \sigma^{\mu\nu}T_{\mu\nu} &= \tfrac{1}{2}\phi^T K_0\phi - \sigma^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}\sigma^\alpha{}_\alpha[(\partial\phi)^2 + m^2\phi^2] \\ &= \tfrac{1}{2}\underbrace{(\delta^{\mu\nu} - 2\sigma^{\mu\nu} + \delta^{\mu\nu}\sigma^\alpha{}_\alpha)}_{\text{shifted metric-like coefficient}}\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}m^2(1+\sigma^\alpha{}_\alpha)\phi^2 \;\equiv\; \tfrac{1}{2}\phi^T K[\sigma]\phi \end{aligned}$$

$K[\sigma] = K_0 + D_1[\sigma]$: the T^2 coupling shifts K_0 by a σ-dependent piece. ϕ now "propagates on σ."

Explicitly on the lattice: $K_0 = -\Delta_{\rm lat} + m^2$, $\quad D_1[\sigma] = (2\sigma^{\mu\nu} - \delta^{\mu\nu}\sigma^\alpha{}_\alpha)\partial_\mu\partial_\nu + m^2\sigma^\alpha{}_\alpha$.

The joint distribution and its two marginals

The joint $p(\phi, \sigma)$ (latent variable model):

$$\boxed{p(\phi, \sigma) \propto e^{-\frac{1}{2}\phi^T K[\sigma]\phi - \frac{1}{2\kappa^2}\sigma_{\mu\nu}\sigma^{\mu\nu}}}$$

(5) Marginalise σ (integrate out latent): $\int d\sigma \; p(\phi, \sigma) = p(\phi)$ in (3). Exact, by HST identity.
(6) Marginalise ϕ (integrate out data): $\int d\phi \; p(\phi, \sigma) = p(\sigma) \propto e^{-S_{\text{eff}}[\sigma]}$. The **emergent** distribution of σ!

(7) Both conditionals Gaussian (like RBM):
Decoder: $p(\phi|\sigma) = \mathcal{N}(0, K[\sigma]^{-1})$ (ϕ is Gaussian given σ; FFT sampling)
Encoder: $p(\sigma(\mathrm{n})|\phi) = \mathcal{N}(\kappa^2 T[\phi],\ \kappa^2)$ (complete the square in σ; site-independent)

σ is the **latent variable** (data augmentation). $p(\phi, \sigma)$ has intractable marginals but tractable conditionals $\rightarrow$ Gibbs sampling.

The effective action: log-density of σ after integrating out ϕ

Integrate out ϕ (a Gaussian with precision $K[\sigma]$):

$$p(\sigma) \propto e^{-S_{\text{eff}}[\sigma]}, \qquad \boxed{S_{\text{eff}}[\sigma] = \underbrace{\frac{1}{2\kappa^2}\sigma^2}_{\text{prior (Gaussian)}} + \underbrace{\tfrac{1}{2}\mathrm{Tr}\ln K[\sigma]}_{\text{from integrating out } \phi}}$$

On the lattice: $\mathrm{Tr}\ln K[\sigma] = \sum_{i=1}^{16} \ln \lambda_i(K[\sigma])$.

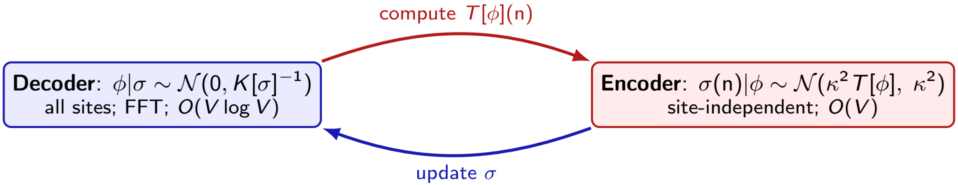
$S_{\text{eff}}[\sigma]$ is **not quadratic** in σ (the Tr log is nonlinear): $p(\sigma)$ is *not* Gaussian.

Two tasks:

- **EM algorithm:** find the *mode* of $p(\sigma)$ $\Leftrightarrow$ minimise $S_{\text{eff}}[\sigma]$ $\Leftrightarrow$ saddle point (Einstein's equation)
- **Gibbs sampler:** *sample* from $p(\sigma)$ $\Leftrightarrow$ include fluctuations around the saddle (quantum gravity)

Gibbs sampler: sampling from $p(\sigma) \propto e^{-S_{\text{eff}}[\sigma]}$

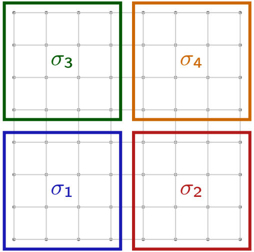
Cannot sample $p(\sigma)$ directly (non-Gaussian). But the **joint** $p(\phi, \sigma)$ has both conditionals Gaussian:



Alternating these two Gaussian steps = Gibbs sampler for the non-Gaussian $p(\sigma)$. No rejection. Exact.

EM = replace encoder with conditional mean (no noise):
M-step: $\bar{\sigma}(\mathrm{n}) = \kappa^2 \langle T \rangle(\mathrm{n}) \ \Leftrightarrow\ \partial S_{\text{eff}}/\partial\sigma = 0 \ \Leftrightarrow\$ **mode** of $p(\sigma)$.
At convergence: Einstein's equation (gap equation).

Blockwise averaging: introducing M



Now coarse-grain the encoder:
Average T over blocks of M sites:
$\bar{T}^B = \frac{1}{M} \sum_{\mathbf{n} \in B} T[\phi](\mathbf{n})$
Set σ constant within each block.

Encoder with block averaging:

$$\sigma_B | \phi \sim \mathcal{N}(\kappa^2 \bar{T}^B, \sim 1/M)$$

Variance $\sim 1/M$: **CLT suppresses quantum**.
M-step: $\bar{\sigma}_B = \kappa^2 \langle T \rangle_B$.

What M controls: quantum mode

$$\boxed{\sigma_B = \underbrace{\kappa^2 \langle T \rangle_B}_{\text{LLN: classical spacetime}} + \underbrace{O(1/\sqrt{M})}_{\text{CLT: quantum gravity}}}$$

Connected correlator $\langle \sigma_B \sigma_{B'} \rangle_c \sim 1/M$: vanishes as $M \to \infty$.

Gravity = most coarse-grained theory in physics ($M \sim 10^{105}$).

System	**Fine $\to$ coarse**	M ($d=3$)
HMM (fluids)	molecule $\to$ cell	$\sim 10^{3\text{–}6}$
Temperature	molecule $\to$ thermometer	$\sim 10^{23}$
Gravity	Planck $\to$ lab	$\sim 10^{105}$

Two approximations: emergent geometry

Define the **metric**: $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} = -2\sigma_{\mu\nu} + O(\sigma^2)$.
The metric tensor $g_{\mu\nu}$ defines the geometry of spacetime through the invariant line element

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu,$$

which remains unchanged under arbitrary coordinate transformations, ensuring that physical distances and causal relations are independent of the chosen coordinates.

Approximation 1 (action of $[\phi|\sigma]$):

$$\boxed{\tfrac{1}{2}\phi^T K[\sigma]\phi = \tfrac{1}{2}\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] + O(\sigma^2)}$$

ϕ propagates on g: the T^2 shift $\approx$ curved spacetime. **Clocks and rulers** (all ϕ-composites) see the same $g \to$ equivalence principle.

Approximation 2 (effective action of σ):

$$\boxed{\mathrm{Tr}\ln K[\sigma] = \mathrm{Tr}\ln \underbrace{\left(-\Box_g + m^2\right)}_{\text{curved Laplacian}} + O(\sigma^2)}$$

Both exact at $O(\sigma)$, with $O(\sigma^2)$ error.

Heat kernel: GR from the curved Laplacian

Expand $\mathrm{Tr}\ln\left(-\Box_g + m^2\right)$ for slowly varying g (Seeley-DeWitt):

$$\frac{a_0}{a^D}\int\sqrt{g} + a_1\int\sqrt{g}\,R + a_2 a^2\int\sqrt{g}\,R^2 + \cdots$$

Term	Scaling	$\lambda = 10a$	Status
Λ (cosmological)	$1/a^D$	10^D	Relevant
$\sqrt{g}R$ (Einstein-Hilbert)	a^0	1	**GR**
$\sqrt{g}R^2$	a^2	10^{-2}	1%
$\sqrt{g}R^3$	a^4	10^{-4}	0.01%

GR = leading non-trivial term in a systematic expansion.

GR is not a fundamental force. It is a macroscopic hydrodynamic mode, like Navier-Stokes. The metric is like temperature: meaningful only after averaging ($M \gg 1$). At the Planck scale ($M \sim 1$), geometry does not exist.

Model 1: from classical Hamiltonian to creation operators

Classical Hamiltonian: $H = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{1}{2}m^2\phi^2$

Quantize: $[\hat{\phi}_{n_x}, \hat{\pi}_{n_y}] = i\delta_{n_x n_y}$. One $(\hat{\phi}, \hat{\pi})$ pair per spatial site (4 pairs on the 4-site lattice).

Diagonalize in momentum (translation $\to$ spectral theorem):

$$\hat{H} = \sum_{k=0}^{3} \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \tfrac{1}{2} \right), \qquad \omega_k = \sqrt{\hat{k}^2 + m^2}$$

Creation/annihilation operators (explicit):

$$\hat{a}_k = \sqrt{\frac{\omega_k}{2}}\hat{\phi}_k + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}_k, \qquad \hat{a}_k^\dagger = \sqrt{\frac{\omega_k}{2}}\hat{\phi}_k - \frac{i}{\sqrt{2\omega_k}}\hat{\pi}_k$$

$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$. $\hat{a}_k^\dagger$: creates a particle with momentum k. $\hat{a}_k$: destroys one.

Fock space: the state space of Model 1

$\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$: counts particles with momentum k.
Fock space = span of all occupation-number states:

$$|n_0, n_1, n_2, n_3\rangle, \qquad n_k = 0, 1, 2, \ldots$$

$|0,0,0,0\rangle = |0\rangle$: vacuum (no particles).
$|1,0,0,0\rangle = \hat{a}_0^\dagger|0\rangle$: one particle at $k = 0$.
$|0,1,1,0\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger|0\rangle$: two particles.

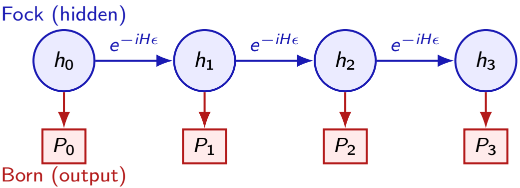
Fock space is **infinite-dimensional** (arbitrarily many particles), even on a 4-site lattice.

Each basis vector $|n_0, n_1, n_2, n_3\rangle$ is **monosemantic**: it has a definite, interpretable meaning (exactly n_k particles at momentum k).
The hidden state h_t is a **polysemantic superposition**:

$$h_t = \sum_{n_0, n_1, n_2, n_3} c_{n_0 n_1 n_2 n_3}(t)\, |n_0, n_1, n_2, n_3\rangle$$

The coefficients $c_{n_0 \cdots n_3}(t)$ are **probability amplitudes**: $|c|^2$ = probability of finding that particle configuration upon measurement.

Model 1 as RNN: hidden state, weight, and output



Hidden layer: $h_t = e^{-iHt} h_0 \in \mathcal{F}$ (Fock space). Weight: $W = e^{-iH\epsilon} = \mathrm{I} - iH\epsilon + \cdots$ (**unitary**).
Output layer (Born rule): measure observable $\hat{O}$ with eigenvalues λ_i and eigenvectors $|o_i\rangle$:

$$P(\lambda_i) = |\langle o_i | h_t \rangle|^2 = \frac{|\langle o_i | h_t \rangle|^2}{\sum_j |\langle o_j | h_t \rangle|^2} \quad \text{(squared-softmax!)}$$

After measurement: **collapse** $h_t \to |o_i\rangle$ (re-embedding into the eigenstate).

Observables, Born rule, and the GPT analogy

Observable $\hat{O}$: Hermitian operator. Spectral decomposition $\hat{O} = \sum_i \lambda_i |o_i\rangle\langle o_i|$.
$|o_i\rangle$ = **readout vectors** (eigenvectors). λ_i = possible outcomes. $P(\lambda_i) = |\langle o_i|h_t\rangle|^2$.
Born rule = squared-softmax: logits $z_i = \langle o_i|h_t\rangle$ (complex inner products), probabilities $P_i = |z_i|^2 / \sum |z_j|^2$.
Collapse = re-embedding: after observing λ_i, $h_t \to |o_i\rangle$ (project, then continue the RNN).

GPT analogy:

	QFT (Model 1)	**GPT**
Hidden state	$h_t \in \mathcal{F}$ (Fock)	$h_t \in \mathbb{R}^d$ (residual stream)
Weight	$e^{-iH\epsilon}$ (unitary)	W (learned, non-unitary)
Readout	$\langle o_i\|h\rangle$ (eigenstate)	$w_i^T h$ (unembedding)
Output	$\|z_i\|^{\mathbf{2}}$ (Born, squared)	softmax(z_i) (exponential)
Collapse	$h \to \|o_i\rangle$ (eigenstate)	$h \to e_i$ (token embedding)

Degeneracy: if $\lambda_i = \lambda_j$ ($i \neq j$), multiple eigenstates give the same outcome. The observer cannot distinguish them $\to$ partial information about h_t.

ϕ^4: the Hamiltonian as polynomial of $\hat{a}, \hat{a}^\dagger$

Add the interaction:

$$\hat{H}_I = \frac{\lambda}{4!}\sum_{n_x}\hat{\phi}^4_{n_x} = \frac{\lambda}{4!}\sum_{n_x}\left(\sum_k \frac{\hat{a}_k e^{ikn_x a} + \hat{a}_k^\dagger e^{-ikn_x a}}{\sqrt{2\omega_k}}\right)^4$$

Expanding: $\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger$ (create 4), $\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$ (scatter $2\to 2$), ..., $\hat{a}\hat{a}\hat{a}\hat{a}$ (destroy 4).
$\hat{H} = \hat{H}_0 + \hat{H}_I$: off-diagonal in Fock basis. Particles scatter.

The RNN update (infinitesimal time step):

$$\boxed{h_{t+dt} = (\mathbb{I} - i\hat{H}\,dt)\,h_t = h_t - i(\hat{H}_0 + \hat{H}_I)\,dt\ h_t}$$

$\hat{H}_0\,dt$: each mode accumulates phase $\omega_k\,dt$ (free propagation).
$\hat{H}_I\,dt$: the polynomial in $\hat{a}, \hat{a}^\dagger$ creates, annihilates, and scatters particles.
Unrolling this RNN over many time steps produces the full story of particle creation, propagation, scattering, and annihilation — all from one matrix-vector multiply per step.

From Hamiltonian to path integral: matrix multiplication

Key identity: $e^{-iHT} = (e^{-iH\epsilon})^{N_t} = \underbrace{W \cdot W \cdot \cdots \cdot W}_{N_t \text{ factors}} \quad (W = e^{-iH\epsilon},\ T = N_t\epsilon)$

Insert resolution of identity $1 = \int d\vec{\phi}\, |\vec{\phi}\rangle\langle\vec{\phi}|$ between each factor:

$$\langle\vec{\phi}_T|e^{-iHT}|\vec{\phi}_0\rangle = \int \prod_{t=1}^{N_t-1} d\vec{\phi}_t \ \ \underbrace{\langle\vec{\phi}_T|W|\vec{\phi}_{N_t-1}\rangle}_{W_{\phi_T,\phi_{N_t-1}}} \cdots \underbrace{\langle\vec{\phi}_2|W|\vec{\phi}_1\rangle}_{W_{\phi_2,\phi_1}} \underbrace{\langle\vec{\phi}_1|W|\vec{\phi}_0\rangle}_{W_{\phi_1,\phi_0}}$$

This IS matrix multiplication: sum over all intermediate $\vec{\phi}_1, \ldots, \vec{\phi}_{N_t-1}$ of the product of matrix elements.

Each matrix element: $W_{\phi',\phi} = \langle\vec{\phi}'|e^{-iH\epsilon}|\vec{\phi}\rangle \approx e^{i\epsilon\mathcal{L}_M(\phi',\phi)} \to$ the path integral $e^{i\frac{1}{2}\vec{\phi}^T K_M \vec{\phi}}$.

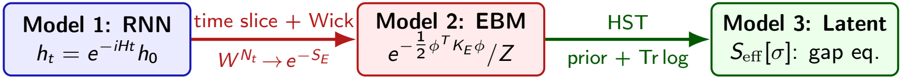
Euclidean version ($t \to -i\tau$): $\hat{T} = e^{-H\epsilon}$ has positive matrix elements.
$\hat{T}_{\phi',\phi} \geq 0$, $\sum_{\phi'} \hat{T}_{\phi',\phi} =$ const: a **Markov chain transition matrix**! The path integral

Closing the circle: Model 1 $\to$ Model 2 $\to$ Model 3

Dyson series (for ϕ^4 interactions): $\hat{U}(t) = T\exp\left(-i\int \hat{H}_I\,dt'\right) = 1 - i\int \hat{H}_I\,dt' + \cdots$

Same Feynman diagrams as the Euclidean Taylor expansion of $e^{-\lambda\phi^4/(4!)}$.

The three models are one theory:



Model 1 $\xrightarrow{\text{matrix mult + Wick}}$ Model 2 $\xrightarrow{\text{HST + integrate out } \phi}$ $S_{\text{eff}}[\sigma]$ $\xrightarrow{\text{saddle}}$ Einstein $\xrightarrow{\text{polish}}$ GR.

Summary: three generative models

	Continuum	4×4 lattice
Model 1: RNN (Lorentzian / Hamiltonian)		
Hamiltonian	$H = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}m^2\phi^2$	**4 oscillators**
Time evolution	$h_t = e^{-iHt} h_0$ **(unitary)**	**Fock-space rotation**
Particles	$a_k^\dagger \lvert 0 \rangle$	**4 creation operators**
Model 2: EBM (Euclidean / exponential family)		
Distribution	$p(\phi) \propto e^{-S_E}$	$\mathcal{N}(0, K_E^{-1})$: **16-dim Gaussian**
Propagator	$G = K_E^{-1} = 1/(k^2 + m^2)$	**Covariance matrix**
Mass	**Decay rate of** $G_E(\tau)$	**Fit exponential**
Model 3: Latent variable (Gravity)		
Saddle (EM, LLN)	$\bar{\sigma} = \kappa^2 \langle T \rangle$ **(Einstein = gap eq.)**	**Deterministic M-step**
Fluctuation (CLT)	$\langle \sigma \sigma \rangle_c \sim 1/M$ **(graviton)**	**Encoder noise** κ^2
Polishing	$h = -2\kappa^2 \langle T \rangle + O(\sigma^2)$	$K[\sigma] \approx K[g]$
Heat kernel	$\sqrt{g}R$ **from** $\operatorname{Tr}\log$	**16 log-eigenvalues**

Einstein = gap equation (Navier-Stokes of gravity). Geometry = emergent (like temperature). At quantum level: non-existent. At strong field: departure from Einstein.

Appendix A: the stress-energy tensor — three perspectives

1. Noether (kinematic): translation invariance $\to \partial^\mu T_{\mu\nu} = 0$.
$T^{\mathrm{can}}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \delta_{\mu\nu}\mathcal{L}$.
Conservation is **kinematic** (from symmetry, regardless of dynamics).
T^2 introduces **dynamics**: how T sources σ (geometry).
Hydrodynamic mode protected by conservation: $\nabla^\mu G_{\mu\nu} = 0$ (Bianchi).

2. Hilbert (variational): $T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_{\mathrm{matter}}}{\delta g^{\mu\nu}}$.
At $g = \delta$: same as canonical. HST coupling $\sigma \cdot T$ = first-order variation of "ϕ propagates on g":
$S[\phi, \delta + h] = S[\phi, \delta] - \frac{1}{2}\int h^{\mu\nu} T_{\mu\nu} + O(h^2)$.

3. Wilson RG: T^2 generated by integrating out high-k modes. $T_{\mu\nu}$ = unique rank-2 conserved current $\to T^2$ = leading rank-2 interaction.

Appendix B: how the $\mathrm{Tr}\log$ arises

Exact (HST): $\int d^V\phi\, e^{-\frac{1}{2}\phi^T K\phi} = (2\pi)^{V/2} e^{-\frac{1}{2}\mathrm{Tr}\ln K}$. On 4×4: $\sum_{i=1}^{16}\ln\lambda_i$.
Approximate (general QFT): expand around saddle $\bar{\phi}$; one-loop $= \frac{1}{2}\mathrm{Tr}\ln S''[\bar{\phi}]$.

Schwinger proper time: $\ln K = -\int_0^\infty \frac{ds}{s}(e^{-sK} - e^{-s})$
$\mathrm{Tr}\, e^{-sK} = \sum_{i=1}^{16} e^{-s\lambda_i}$ (16 decaying exponentials)

Why "heat kernel": e^{-sK} solves $\partial_s u = -Ku$ with $u(0) = \delta$. If $K = -\Delta + m^2$: literal heat diffusion. $s =$ fictitious diffusion time.

Small-s expansion: $\mathrm{Tr}\, e^{-sK} \sim \frac{1}{(4\pi s)^{D/2}} \sum_n a_n\, s^n$ (Seeley-DeWitt coefficients)

Appendix B: unpolished $\leftrightarrow$ polished correspondence

n	Unpolished $a_n[\sigma]$	Polished $a_n[g]$	Connection
0	$\sum f_0(\sigma)$	$\int \sqrt{g}$	$\sqrt{g} \approx 1 - \sigma^\alpha{}_\alpha$
1	$\sum[c\Delta\sigma + \cdots]$	$\frac{1}{6} \int \sqrt{g} R$	$R \sim \partial^2\sigma$
2	$\sum[(\Delta\sigma)^2 + \cdots]$	$\frac{1}{180} \int \sqrt{g}(R^2_{\mu\nu\rho\sigma} - R^2_{\mu\nu} + \cdots)$	$R^2 \sim (\partial^2\sigma)^2$

Polishing = invertible change of variables (at weak field).
Unpolished $a_n[\sigma]$: computable on the lattice (sums over 16 sites).
Polished $a_n[g]$: same numbers in geometric language.
At strong field: correspondence breaks down; lattice remains valid.

GR example: $\frac{M}{2} a_1 \int \sqrt{g} R = \frac{1}{16\pi G_N} \int \sqrt{g} R \;\Rightarrow\; G_N^{-1} \propto M a_1$.
Gauge example: a_1 of $-D_\mu^2 + m^2$ gives $\int \mathrm{tr}(F^2)$ (Maxwell/YM).
Same mechanism at every rank: Gaussian prior + matter loops $\to$ induced kinetics.

Appendix C: absorbing the prior (condensed matter practice)

Saddle: $\bar{\sigma}/\kappa^2 + \langle T \rangle_{\bar{\sigma}} = 0 \;\Rightarrow\; \bar{\sigma} = -\kappa^2 \langle T \rangle$.
At saddle: $\bar{\sigma}^2/(2\kappa^2) = \kappa^2 \langle T \rangle^2/2$ — known functional of $\mathrm{Tr}\log$. **No independent information.**

Condensed matter recipe:

1. Define dressed propagator $G = (K_0 + \bar{\sigma} \cdot V)^{-1}$.
2. Gap equation $\bar{\sigma} = -\kappa^2 \langle T \rangle_G$ determines G.
3. Prior absorbed into metric $g_{\mu\nu} = \delta_{\mu\nu} - 2\bar{\sigma}_{\mu\nu}$.

Only need the saddle of the $\mathrm{Tr}\log$. Prior sets coupling κ^2, then disappears into g.

Appendix C: extracting the dressed propagator from Gibbs

Gibbs samples $(\phi^{(s)}, \sigma^{(s)})$ from $p(\phi, \sigma)$.

Quantity	**From samples**	**Physics**
Dressed propagator	$G_{\mathbf{nm}} = \frac{1}{S}\sum_s \phi_{\mathbf{n}}^{(s)}\phi_{\mathbf{m}}^{(s)}$	Classical metric (LLN)
Sample mean $\bar{\sigma}$	$\frac{1}{S}\sum_s \sigma^{(s)}$	Saddle = EM output
Sample variance	$\mathrm{Var}(\sigma) \sim 1/M$	Quantum corrections (CLT)

EM gives the mode (classical GR).
Gibbs gives mode + fluctuations (GR + quantum gravity).
After polishing: $G \to (-\Box_g + m^2)^{-1}$, heat kernel $\to \sqrt{g}R$.
GR comes entirely from $\mathrm{Tr}\log$. Prior is gone.