Directed Mixed Graphs for Latent Variables

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Stats 212 Graphical Models
Lecture Notes

Outline

Acyclic directed mixed graphs

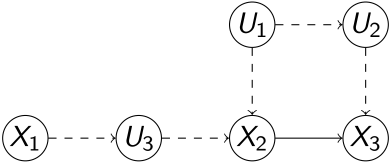
Latent projection of a DAG (Tian and Pearl 2002b):
Given a DAG with latent variables $\mathcal{G}(V \cup L)$, where V is observed and L latent, the *latent projection* $\mathcal{G}(V)$ is constructed as follows:

1. $\mathcal{G}(V)$ contains an edge $a \to b$ if there is a directed path $a \to \cdots \to b$ in $\mathcal{G}(V \cup L)$ with all intermediate vertices in L.
2. $\mathcal{G}(V)$ contains an edge $a \leftrightarrow b$ if there is a collider-free path $a \leftarrow \cdots \to b$ with all intermediate vertices in L.

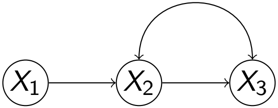
Note: Step 1 adds all directed edges $a \to b$ in $\mathcal{G}(V \cup L)$ to $\mathcal{G}(V)$.

Acyclic directed mixed graphs

DAG $\mathcal{G}(V \cup L)$, $V = \{X_1, X_2, X_3\}$ and $L = \{U_1, U_2, U_3\}$:



Latent projection $\mathcal{G}(V)$ is an acyclic directed mixed graph (ADMG):



Acyclic directed mixed graphs

Definitions. Let $\mathcal{G} = (V, E)$ be a directed mixed graph, i.e. a graph with two types of edges: directed ($\rightarrow$) or bidirected ($\leftrightarrow$).

- A *path* is a sequence of distinct adjacent edges, of any type or orientation, between distinct vertices.
  directed path: $a \rightarrow \cdots \rightarrow b$. bidirected path: $a \leftrightarrow \cdots \leftrightarrow b$.
- If $a \rightarrow b$, then a is a parent of b and b is a child of a.
- If there is a directed path from a to d or $a = d$, we say a is an ancestor of d and d is a descendant of a. Accordingly define non-descendant.
- If $a \leftrightarrow b$, then a is a sibling of b.
- notation: $\mathrm{pa}_{\mathcal{G}}(a)$, $\mathrm{ch}_{\mathcal{G}}(a)$, $\mathrm{an}_{\mathcal{G}}(a)$, $\mathrm{de}_{\mathcal{G}}(a)$, $\mathrm{nd}_{\mathcal{G}}(a)$, and $\mathrm{sib}_{\mathcal{G}}(a)$.

Acyclic directed mixed graphs

- A *directed cycle* is a path of the form $v \to \cdots \to w$ along with an edge $w \to v$.
- An acyclic directed mixed graph (ADMG) is a mixed graph containing no directed cycles.
- A topological sort of an ADMG is defined in the same way as for a DAG: $a \to b$ implies $a \prec b$.

Acyclic directed mixed graphs

m-separation:

- A vertex z is a collider on a path if $\rightarrow z \leftarrow$, $\leftrightarrow z \leftrightarrow$, $\rightarrow z \leftrightarrow$, or $\leftrightarrow z \leftarrow$; otherwise, z is a non-collider.
- *m*-connection: A path between a and b is *m*-connecting given C if (i) every non-collider on the path is not in C and (ii) every collider on the path is an ancestor of C ($\text{an}(C) := \cup_{a \in C} \text{an}(a)$).
- *m*-separation: If there is no path *m*-connecting a and b given C, then a and b are *m*-separated given C.
- If $\mathcal{G}$ is a DAG, *m*-separation is identical to *d*-separation.

Acyclic directed mixed graphs

m-separation:

Proposition 1 (Richardson et al. (2023))

Let $\mathcal{G}(V \cup L)$ be a DAG and $\mathcal{G}(V)$ be its latent projection. For disjoint subsets $A, B, C \subset V$, A and B are *d*-separated given C in $\mathcal{G}(V \cup L)$ if and only if A and B are *m*-separated given C in $\mathcal{G}(V)$.

- On every path between $a, b \in V$ in $\mathcal{G}(V \cup L)$, colliders (resp. non-colliders) in V are also colliders (resp. non-colliders) on a path in $\mathcal{G}(V)$.
- ADMG $\mathcal{G}(V)$ captures all conditional independence constraints among the observed variables V in the DAG $\mathcal{G}(V \cup L)$ with latent variables.

Acyclic directed mixed graphs

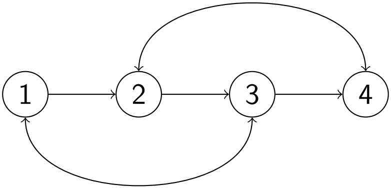
Districts in ADMG $\mathcal{G}(V)$:

- The *district* of vertex v, denoted $\text{dis}_{\mathcal{G}}(v)$, is the set of vertices that are connected to v by a bidirected path (including v itself).
- A district of $\mathcal{G}$ is a maximal bidirected-connected set of vertices.
- A district corresponds to a confounded component (c-component) (Tian and Pearl 2002b).
- Districts specify variable partitions that define terms in the factorization of $\mathbb{P}(V)$.

Denote districts by $\mathcal{D}(\mathcal{G}) = \{D : D \text{ is a district of } \mathcal{G}\}$.
Define $\text{pa}_{\mathcal{G}}(D) := (\cup_{a \in D} \text{pa}_{\mathcal{G}}(a)) \setminus D$.

Factorizations on ADMGs

District factorization:



- Districts of $\mathcal{G}$: $D_1 = \{1,3\}, D_2 = \{2,4\}$.
- $\text{pa}_{\mathcal{G}}(D_1) = \{2\}$, $\text{pa}_{\mathcal{G}}(D_2) = \{1,3\}$.

Using $a \leftrightarrow b \Leftrightarrow a \leftarrow u \rightarrow b$:

$$p(x_1,\ldots,x_4) = \left[\sum_{u_1} p(x_1 \mid u_1)p(x_3 \mid x_2, u_1)p(u_1)\right] \times \left[\sum_{u_2} p(x_2 \mid x_1, u_2)p(x_4 \mid x_3, u_2)p(u_2)\right]$$
$$= q_{1,3}(x_1, x_3 \mid x_2) \times q_{2,4}(x_2, x_4 \mid x_1, x_3).$$

Factorizations on ADMGs

$$p(x_1, \ldots, x_4) = q_{1,3}(x_1, x_3 \mid x_2) \times q_{2,4}(x_2, x_4 \mid x_1, x_3)$$
$$= q_{D_1}(x_{D_1} \mid \mathrm{pa}_{\mathcal{G}}(D_1)) \times q_{D_2}(x_{D_2} \mid \mathrm{pa}_{\mathcal{G}}(D_2)).$$

For general case, district factorization:

$$\mathbb{P}(V) = \prod_{D \in \mathcal{D}(\mathcal{G})} q_D(x_D \mid \mathrm{pa}_{\mathcal{G}}(D)). \tag{1}$$

- Each factor $q_Y(y \mid W)$ is called a *kernel*, i.e. a probability density of Y with W being a parameter: $\sum_y q_Y(y \mid W = w) = 1$, $\forall w$.
- $q_Y(y \mid W = w) = \mathbb{P}(Y = y \mid do(w))$ and thus, in general $q_Y(y \mid W) \neq \mathbb{P}(Y = y \mid W = w)$.

Factorizations on ADMGs

Express $q_D(x_D \mid \mathrm{pa}_{\mathcal{G}}(D))$ as $\prod_{i \in D} p(x_i \mid \cdots)$:

- The Markov blanket of $a \in V$ in ADMG $\mathcal{G}$ is

$$\mathrm{mb}(a, \mathcal{G}) := \mathrm{pa}_{\mathcal{G}}(D) \cup (D \setminus \{a\}),$$

  where $D = \mathrm{dis}_{\mathcal{G}}(a)$. We have $a \perp \mathrm{nd}(a) \mid \mathrm{mb}(a)$.

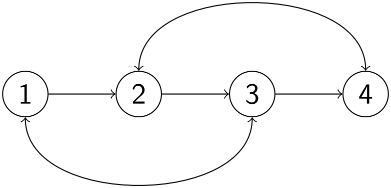
- Suppose that $1 \prec \cdots \prec p = |V|$ is a topological sort of $\mathcal{G}$. Let $V_i = \{1, \ldots, i\}$ and $\mathcal{G}_i$ be the induced subgraph on V_i. Then $X_i \perp X_k \mid \mathrm{mb}(i, \mathcal{G}_i)$, $k < i$:

$$q_D(x_D \mid \mathrm{pa}_{\mathcal{G}}(D)) = \prod_{i \in D} p(x_i \mid \mathrm{mb}(i, \mathcal{G}_i)). \tag{2}$$

- Putting together into (1), we get

$$\mathbb{P}(V) = \prod_{i \in V} p(x_i \mid \mathrm{mb}(i, \mathcal{G}_i)). \tag{3}$$

Factorizations on ADMGs



Sort: $1 \prec 2 \prec 3 \prec 4$.
$\mathrm{mb}(1, \mathcal{G}_1) = \varnothing$,
$\mathrm{mb}(2, \mathcal{G}_2) = \{1\}$,
$\mathrm{mb}(3, \mathcal{G}_3) = \{1, 2\}$,
$\mathrm{mb}(4, \mathcal{G}_4) = \{1, 2, 3\}$.

$$q_{1,3}(x_1, x_3 \mid x_2) = p(x_1)p(x_3 \mid x_1, x_2), \quad (4)$$

$$q_{2,4}(x_2, x_4 \mid x_1, x_3) = p(x_2 \mid x_1)p(x_4 \mid x_1, x_2, x_3). \quad (5)$$

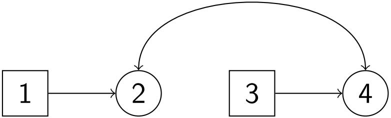
$$\Rightarrow \; p(x) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2)p(x_4 \mid x_1, x_2, x_3).$$

This does NOT imply any conditional independence among $X_1, \cdots, X_4$.
In particular, $X_1 \not\perp X_4 \mid S$ for any $S \subseteq \{X_2, X_3\}$ (m-connected) even though no edge between X_1 and X_4.

Generalized CI constraints

No edge between X_1 and X_4 encodes a generalized conditional independence a.k.a. Verma constraint (Verma and Pearl 1990).

Represent $q_{2,4}(x_2, x_4 \mid x_1, x_3) = p(x_2, x_4 \mid do(x_1, x_3))$ by a conditional ADMG (CADMG) with graph $\mathcal{G}^{|W}$ ($W = \{1, 3\}$) by cutting all edges with an arrow into W:



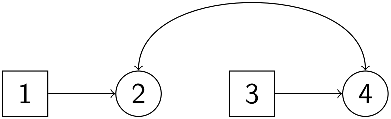
- Two types of vertices in a CADMG $\mathcal{G}(V, W)$:
  (i) Random $V = \{2, 4\}$; (ii) Fixed $W = \{1, 3\}$.
- Kernel $q_V(x_V \mid x_W)$ is an (intervention) distribution for V after fixing W.
- We may further fix other random vertices if they are *fixable*.

Generalized CI constraints

Definition 1

The set of *fixable* vertices in a CADMAG $\mathcal{G}(V, W)$ is $F(\mathcal{G}) := \{v \in V : \mathrm{dis}_{\mathcal{G}}(v) \cap \mathrm{de}_{\mathcal{G}}(v) = \{v\}\}$.

v is fixable if none of its descendants is in the same district.



Fixable vertices $=\{2, 4\}$.

Fix vertex 2: (i) $\mathcal{G}(V = \{4\}, W = \{1, 2, 3\})$

(ii) New kernel district-factorized according to $\mathcal{G}(\{4\}, \{1, 2, 3\})$:

$$q_4(x_4 \mid x_2, x_1, x_3) = f_4(x_4 \mid x_3). \quad \text{nested factorization} \tag{6}$$

Generalized CI constraints

The new kernel $q_4(x_4 \mid x_2, x_1, x_3)$ is defined by the fixing operator:

Definition 2

Given a kernel $q_V(x_V \mid W)$ associated with a CADMG $\mathcal{G} = \mathcal{G}(V, W)$, for any fixable vertex $r \in F(\mathcal{G})$, the fixing operator ϕ_r yields a new kernel

$$q_{V\setminus r}(x_{V\setminus r} \mid r, W) = \phi_r(q_V; \mathcal{G}) := \frac{q_V(x_V \mid W)}{q_V(x_r \mid \mathrm{mb}(r, \mathcal{G}), W)}. \quad (7)$$

- $q_V(x_r \mid \mathrm{mb}(r, \mathcal{G}), W)$ is a conditional distribution calculated from $q_V(x_V \mid W)$.
- If r is fixable, then r can be sorted as the last vertex in its district and its causal effect $\mathbb{P}(V \setminus r \mid do(r); \mathcal{G})$ on $V \setminus r$ can be calculated by (7).

Generalized CI constraints

Apply ϕ_2 on $q_{2,4}(x_2, x_4 \mid x_1, x_3)$ ($\text{mb}(2, \mathcal{G}) = \{1, 4, 3\}$):

$$\begin{aligned} q_4(x_4 \mid x_2, x_1, x_3) &= \phi_2(q_{2,4}; \mathcal{G}) = \frac{q_{2,4}(x_2, x_4 \mid x_1, x_3)}{q_{2,4}(x_2 \mid x_4, x_1, x_3)} \\ &= q_{2,4}(x_4 \mid x_1, x_3) \\ &= \sum_{x_2'} q_{2,4}(x_2', x_4 \mid x_1, x_3) \\ &= \sum_{x_2'} p(x_2' \mid x_1) p(x_4 \mid x_1, x_2', x_3). \quad \text{by (5)} \end{aligned}$$

By nested factorization (6):

$$\sum_{x_2'} p(x_2' \mid x_1) p(x_4 \mid x_1, x_2', x_3) = f_4(x_4 \mid x_3)$$

does not depend on x_1, which is a GCI constraint.

Generalized CI constraints

Nested factorization:

- Suppose $p(x)$ factorizes by a DAG $\mathcal{G}(V \cup L)$ and $\mathcal{G} = \mathcal{G}(V)$ is the ADMG defined by latent projection.
- For a valid fixing sequence $w = (w_1, \ldots, w_r)$, let $\phi_w(\mathcal{G})$ be the CADMG after fixing w sequentially and $\mathcal{D}_w = \mathcal{D}(\phi_w(\mathcal{G}))$ be the districts of (random vertices) in $\phi_w(\mathcal{G})$.

Theorem 1 (Richardson et al. (2023))

For any valid fixing sequence w,

$$\phi_w(p(x_V); \mathcal{G}) = \prod_{D \in \mathcal{D}_w} f_D^w(x_D \mid \mathrm{pa}_{\mathcal{G}}(D))$$

for some kernels $f_D^w(x_D \mid \mathrm{pa}_{\mathcal{G}}(D))$.

Generalized CI constraints

Algorithm to find systematically CI and GCI constraints implied by ADMG: Tian and Pearl (2002b).

Input: ADMG $\mathcal{G}(V)$; assume V is sorted, $1 \prec \ldots \prec p$.
Output: CI and GCI constraints on $p(x_V)$ implied by $\mathcal{G}(V)$.

For $i = 1$ to p,
Part 1: CI constraints $X_i \perp X_k \mid \mathrm{mb}(i, \mathcal{G}_i)$, $k < i$, $k \notin \mathrm{mb}(i, \mathcal{G}_i)$.
Part 2: $S \leftarrow \mathrm{dis}_{\mathcal{G}_i}(i)$ and $G \leftarrow \phi_{[i]\setminus S}(\mathcal{G}_i)$ ($[i] = \{1, \ldots, i\}$).
For each descendent set $D \subset S$ s.t. $i \notin D$: Let $D' = S \setminus D$.

1. $\sum_{x_D} q_S = q_{D'}$ (fixing D); $G' = \phi_D(G)$.
2. If G' has 2 or more districts, $E \leftarrow \mathrm{dis}_{G'}(i)$ and $q_{D'} / \sum_{x_i} q_{D'}$ is a function of $\mathrm{mb}(i, G') = E \cup \mathrm{pa}_{G'}(E)$.
3. Repeat part 2 with $S \leftarrow E$ and $G \leftarrow \phi_{S \setminus E}(G)$.

Identification of causal effects

Identification of causal effects given an ADMG $\mathcal{G}(V)$:

- Let $k \in V$ be a single variable and $S \subset V$.
- The causal effect of X_k on S is identifiable (from observational data) if $\mathbb{P}(S \mid do(X_k))$ can be computed from the joint distribution $\mathbb{P}(V)$.

Theorem 2 (Tian and Pearl (2002a))

If there is no bidirected path connecting X_k to any of its children in $\mathcal{G}_{\mathrm{an}(S)}$, then the causal effect of X_k on S is identifiable.

- Recent results: Theorem 48 in Richardson et al. (2023), Corollary 16 in Bhattacharya et al. (2022).

Identification of causal effects

Constructive proof of Theorem 2:

1. Let $V = \text{an}(S)$, $\mathcal{G} = \mathcal{G}_{\text{an}(S)}$ and $M = V \setminus \{S \cup k\}$. Then

$$p(x_S \mid do(x_k)) = \sum_{x_M} p(x_{V\setminus k} \mid do(x_k)).$$

2. Let $D = \text{dis}_{\mathcal{G}}(k) \in \mathcal{D} = \mathcal{D}(\mathcal{G})$. Since $\text{ch}(k) \cap D = \varnothing$,

$$p(x_{V\setminus k} \mid do(x_k)) = \sum_{x'_k} q_D(x_D \mid \text{pa}_{\mathcal{G}}(D)) \prod_{D' \in \mathcal{D}} q_{D'}(x_{D'} \mid \text{pa}_{\mathcal{G}}(D')).$$

If X_k is fixable, we may instead apply fixing operator:

$$p(x_{V\setminus k} \mid do(x_k)) = \phi_k(p(x); \mathcal{G}) = \frac{p(x_V)}{p(x_k \mid \text{mb}(k, \mathcal{G}))}.$$

Identification of causal effects

The identify algorithm by Tian and Pearl (2002a) reformulated with fixing operators: Theorem 48 in Richardson et al. (2023).

Let $\mathcal{G} = \mathcal{G}(V)$. For $A, Y \subset V$, want to identify $\mathbb{P}(Y \mid do(A))$.

- Let $Y^* = \text{an}_{\mathcal{G}_{V \setminus A}}(Y) \supseteq Y$: there is a directed path from every $v \in Y^*$ to Y not blocked by A.

  Since $V \setminus (A \cup Y) = [V \setminus (A \cup Y^*)] \cup (Y^* \setminus Y)$,

$$
\begin{aligned}
\mathbb{P}(Y \mid do(A)) &= \sum_{V \setminus (A \cup Y)} \mathbb{P}(V \setminus A \mid do(A)) \\
&= \sum_{Y^* \setminus Y} \sum_{V \setminus (A \cup Y^*)} \mathbb{P}(V \setminus A \mid do(A)) \\
&= \sum_{Y^* \setminus Y} \mathbb{P}(Y^* \mid do(A)), \quad (Y^* \text{ is ancestral}).
\end{aligned}
$$

Identification of causal effects

- Let $\mathcal{D}^* = \mathcal{D}(\mathcal{G}_{Y^*})$. District factorization on $\mathcal{G}_{Y*}$ shows

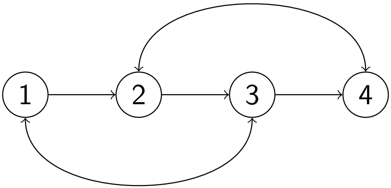
$$\mathbb{P}(Y^* \mid do(A)) = \prod_{D \in \mathcal{D}^*} \mathbb{P}[D \mid do(\mathrm{pa}_{\mathcal{G}}(D))].$$

- If every D is intrinsic (i.e. $V \setminus D$ is fixable), then $\mathbb{P}[D \mid do(\mathrm{pa}_{\mathcal{G}}(D))] = \phi_{V \setminus D}(\mathbb{P}(V); \mathcal{G})$, and

$$\therefore \quad \mathbb{P}(Y \mid do(A)) = \sum_{Y^* \setminus Y} \prod_{D \in \mathcal{D}^*} \phi_{V \setminus D}(\mathbb{P}(V); \mathcal{G}). \qquad (8)$$

  Otherwise, $\mathbb{P}[D \mid do(\mathrm{pa}_{\mathcal{G}}(D))]$ is not identifiable for some D, and $\mathbb{P}(Y \mid do(A))$ is not identifiable.

Identification of causal effects



Find $p(x_4 \mid do(x_2))$.
$Y = \{4\}, A = \{2\}$
$Y^* = \{3, 4\}$
$\mathcal{D}^* = \{D_1, D_2\} = \{\{3\}, \{4\}\}$

$$p(x_3 \mid do(x_2)) = \phi_{1,2,4}(p(x_V); \mathcal{G}) = \phi_1(q_{1,3}(x_1, x_3 \mid x_2); \mathcal{G}^{|2,4})$$
$$= \sum\nolimits_{x_1} p(x_1) p(x_3 \mid x_1, x_2).$$
$$p(x_4 \mid do(x_3)) = \phi_{2,1,3}(p(x_V); \mathcal{G}) = \phi_2(q_{2,4}(x_2, x_4 \mid x_1, x_3); \mathcal{G}^{|1,3})$$
$$= \sum\nolimits_{x_2'} p(x_2' \mid x_1) p(x_4 \mid x_1, x_2', x_3).$$

$$\therefore \quad p(x_4 \mid do(x_2)) = \sum\nolimits_{x_3} p(x_3 \mid do(x_2)) p(x_4 \mid do(x_3)).$$

Linear SEM associated with ADMG

Given an ADMG $\mathcal{G}$ with directed edge set E_d and bidirected edge set E_b, define linear SEM

$$X_j = \sum_{i \in \mathrm{pa}_{\mathcal{G}}(j)} \beta_{ij} X_i + \varepsilon_j, \quad j = 1, \ldots, p. \tag{9}$$

$$(\varepsilon_1, \ldots, \varepsilon_p) \sim \mathcal{N}_p(0, \Omega).$$

- $B \in \mathcal{B}(E_d) := \{(\beta_{ij})_{p \times p} : \beta_{ij} = 0 \text{ if } i \to j \notin E_d\}$.
- $\Omega \in \mathcal{P}(E_b) := \{(\omega_{ij})_{p \times p} : \omega_{ij} = 0 \text{ if } i \leftrightarrow j \notin E_b\}$.

The linear SEM (9) defines a family of multivariate Gaussian distributions $\mathcal{N}_p(0, \Sigma)$ with

$$\Sigma = \Sigma_{\mathcal{G}}(B, \Omega) := (\mathbf{I} - B)^{-\mathsf{T}} \Omega (\mathbf{I} - B)^{-1}.$$

Linear SEM associated with ADMG

Definition 3 (Identifiability)

The linear SEM for an ADMG $\mathcal{G}$ is said to be identifiable if $\Sigma_{\mathcal{G}}(B, \Omega)$ is an *injective* (one-to-one) map from $\mathcal{B}(E_d) \times \mathcal{P}(E_b)$ to the set of positive definite matrices.

Reachable closure (Shpitser et al. 2018).

Definition 4

For a CADMG $\mathcal{G}(V, W)$, a reachable subset $C \subseteq V$ is called a reachable closure for $S \subseteq C$ if the set of fixable vertices in $\phi_{V \setminus C}(\mathcal{G})$ is a subset of S.

- Reachable closure is unique for any $S \subseteq V$, denoted $\langle S \rangle$.
- $\langle S \rangle$ is the set of random vertices in $\phi_{\neg S}(\mathcal{G})$ (fixing as many vertices in $V \setminus S$ as possible).

Linear SEM associated with ADMG

Graphical criterion for identifiability:

Theorem 3 (Drton et al. (2011))

The linear SEM for an ADMG $\mathcal{G}(V)$ is identifiable if and only if $\langle v \rangle = \{v\}$ for all $v \in V$.

- Identifiability means that given $\mathcal{G}(V)$ and Σ, there is a unique set of parameters (B, Ω) for the linear SEM. Thus, given $\mathcal{G}(V)$ and data, one may estimate (B, Ω).
- Example: $a \rightarrow s \leftarrow b$ and $b \leftrightarrow a \leftrightarrow s$.
  - $\langle s \rangle = \{a, b, s\}$ (a, b are not fixable in $V \setminus s$).
  - $\mathcal{G}_{a,b,s}$ contains a sink node s and its parents a, b in the same district.
  - Linear SEM is *not* identifiable.

Ancestral graphs

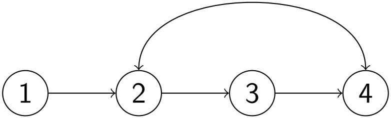
Motivations.

- A class of ADMGs that represents conditional independences among V in a DAG $\mathcal{G}(V, L)$ with latent variables L.
- Retains the ancestral relationships and hence causal relations among V.
- Its equivalence class can be constructed from CI relations learned from observational data.
- Does *not* preserve all confounding structures in $\mathcal{G}(V, L)$, i.e. bidirected edges in the latent projection.
- Does *not* represent GCI (Verma) constraints: potential loss of efficiency.

Ancestral graphs

Definitions. Let $\mathcal{G} = (V, E)$ be an ADMG.

- An *almost directed cycle* occurs when $a \leftrightarrow b$ and $a \in \mathrm{an}_{\mathcal{G}}(b)$ (removing the arrowhead at b results in a directed cycle).
- Let $L \subset V$. An *inducing path relative to L* is a path on which every intermediate vertex $\notin L$ is a collider and every collider is an ancestor of an endpoint. If $L = \varnothing$, call it an inducing path.



Almost directed cycle: $(2, 3, 4, 2)$.
Inducing path: $1 \to 2 \leftrightarrow 4$
$\Rightarrow$ 1 and 4 not *m*-separated by any subsets.

Ancestral graphs

Definition 5 (MAG)

A mixed graph is a maximal ancestral graph (MAG) if

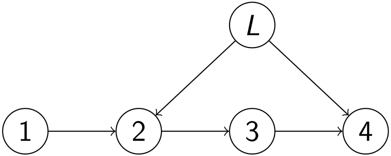
(i) it does not contain any directed or almost directed cycles (ancestral);

(ii) there is no inducing path between any two non-adjacent vertices (maximal).

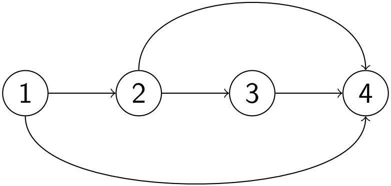
Constructing MAG $\mathcal{M}$ from DAG $\mathcal{G} = \mathcal{G}(V, L)$:

1. For each pair $a, b \in V$, a and b are adjacent in $\mathcal{M}$ iff there is an inducing path between them relative to L in $\mathcal{G}$.
2. For each adjacent pair (a, b) in $\mathcal{M}$,
   orient $a \to b$ in $\mathcal{M}$ if $a \in \text{an}_{\mathcal{G}}(b)$; orient $b \to a$ in $\mathcal{M}$ if $b \in \text{an}_{\mathcal{G}}(a)$; orient $a \leftrightarrow b$ otherwise.

DAG $\mathcal{G}(V \cup L)$



MAG



Every edge among V in a DAG (trivial inducing path) is an edge in MAG.

Inducing paths relative to L:

$1 \to 2 \leftarrow L \to 4 \Rightarrow 1 \to 4$ in $\mathcal{M}$

$2 \leftarrow L \to 4 \Rightarrow 2 \to 4$ in $\mathcal{M}$

$1, 2$ are ancestors of 4.

Ancestral graphs

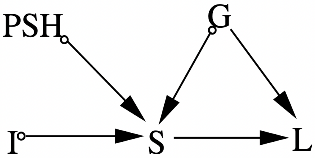
Equivalence class of a MAG:

- Two MAGs are Markov equivalent if they have the same set of *m*-separations.
  Sufficient and necessary conditions: same skeleton and *v*-structures, and share some covered colliders (Proposition 2, Zhang (2008b)).
- The equivalence class $[\mathcal{M}]$ of a MAG $\mathcal{M}$ is represented by a partial ancestral graph $\mathcal{P}$:
  - i $\mathcal{P}$ has the same adjacencies (skeleton) as $\mathcal{M}$;
  - ii A mark of arrowhead is in $\mathcal{P}$ iff it is shared by all MAGs in $[\mathcal{M}]$;
  - iii A mark of tail is in $\mathcal{P}$ iff it is shared by all MAGs in $[\mathcal{M}]$.

  Edge marks in (ii) and (iii) are invariant across $[\mathcal{M}]$; other variable marks are represented by $\circ$ in $\mathcal{P}$.

Ancestral graphs

Example PAG (Zhang 2008a)



I: income, S: smoking, PSH: parent smoking habits, G: genotype, L: lung cancer

- $I \circ\!\!\rightarrow S = I \rightarrow S$ or $I \leftrightarrow S$.
- preserve the 3 v-structures at the collider S.
- no directed or almost directed cycles among G, S, L.

The FCI algorithm

Constraint-based learning of MAGs by the FCI (fast causal inference) algorithm (Spirtes et al. 1999):

Use CI constraints learned from observational data to construct the equivalence class of a MAG represented by a PAG:

- skeleton;
- invariant marks (arrowheads and tails).

The FCI algorithm

Algorithm outline

1. $E \leftarrow$ edge set of the complete undirected graph on V. Every edge is $\circ\!-\!\circ$.
2. **for** $(i, j) \in E$ **do**
3. Search for a subset S_{ij} such that $X_i \perp X_j \mid S_{ij}$. If found, $E \leftarrow E \setminus \{(i, j), (j, i)\}$ and store S_{ij}.
4. **end for**
5. Orient edges in v-structures based on E and $\{S_{ij}\}$.
6. Apply orientation rules R1 to R4 (Zhang 2008b) until none of them applies.
7. Apply orientation rules R8 to R10 (Zhang 2008b) until none of them applies.

The FCI algorithm

Suppose $\mathcal{M}$ is the true MAG, and assume we have CI oracle.

- Line 1 to 5: similar to the PC algorithm.
- After Line 4: correctly construct the skeleton $sk(\mathcal{M})$.
- After Line 6: identify all and only invariant arrowheads in $[\mathcal{M}]$.
- After Line 7: identify all and only invariant tails in $[\mathcal{M}]$.

Theorem 4 (Theorem 4, Zhang (2008b))

Given a perfect conditional independence oracle, the FCI algorithm returns the PAG for the true MAG $\mathcal{M}$.

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