A Non-Measurable Set

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Both Royden[1] (pg 53) and Rudin[2] (pg 54) offer similar constructions/proofs for a non-Lebesgue-measurable set (that they name P and E respectively). The following is a take on their explanations.

The Axiom of Choice: Constructing $\mathring{\Upsilon}$, an Unmeasurable Set

Let us call the set of rationals from the interval $[-1,1]$, $Q_{[-1,1]}$, the set of irrationals from $[1,2)$, $I_{[1,2)}$, and we'll call the set of irrationals from $[0,3)$, $I_{[0,3)}$.

Initial Step: We "seed" $\mathring{\Upsilon}$ with some number in $I_{[1,2)}$.

(...the following step is to be iterated uncountably...)

Iterated Step: Let $x \in I_{[1,2)}$, and let y be **any current** member of $\mathring{\Upsilon}$. If $x - y \notin \mathbb{Q}$, then x is admitted into $\mathring{\Upsilon}$. (Else, if $y - x \in \mathbb{Q}$, then x is excluded from $\mathring{\Upsilon}$.)

(...the above step is repeated testing every $x \in I_{[1,2)}$...) [3]

We now prove $\mathring{\Upsilon}$ is non-Lebesgue-measurable in much the way of Royden and Rudin.

Let $q_i, q_j \in \mathbb{Q}$. By our careful construction of $\mathring{\Upsilon}$, it should be clear that if $i \neq j$, then $(q_i + \mathring{\Upsilon}) \bigcap (q_j + \mathring{\Upsilon}) = \emptyset$. That is, each of these rational "offsets" of *halo-Upsilon* are pairwise disjoint. Furthermore, if we say $\bigcup_k^\infty q_k = Q_{[-1,1]}$, then we have $I_{[1,2)} \subset \bigcup_k^\infty (q_k + \mathring{\Upsilon}) \subset I_{[0,3)}$ (*). The resulting contradiction is fairly intuitive: If $m\mathring{\Upsilon} = 0$, then $mI_{[1,2)} \leq 0 \leq mI_{[0,3)}$. If $m\mathring{\Upsilon} > 0$, then $mI_{[1,2)} \leq \infty \leq mI_{[0,3)}$.

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Proof of (*): That the union of the offsets is a subset of $I_{[0,3)}$ is trivial, so we need show only that $I_{[1,2)}$ is a subset of the union. Let $x \in I_{[1,2)}$ be arbitrary. If $x \in \mathring{\Upsilon}$, then $x \in 0 + \mathring{\Upsilon}$. If $x \notin \mathring{\Upsilon}$, then x must have been excluded from $\mathring{\Upsilon}$ during construction because it differed from a pre-existing member of $\mathring{\Upsilon}$ by some rational number, $q_x \in Q_{[-1,1]}$. Hence $x \in \pm q_x + \mathring{\Upsilon}$.

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[1] Royden, H.L. *Real Analysis*, Macmillan, NY. 1965.

[2] Rudin, Walter. *Real and Complex Analysis*. McGraw-Hill, Singapore. 1987.

[3] Stretching notation we may say, $\mathring{\Upsilon} = \{x : x \in I_{[1,2)},\ y \in \mathring{\Upsilon} \Longrightarrow x - y \notin \mathbb{Q}\}$