STAT 251 / OBEE 216
Winter 2003
Prof. Ivo D. Dinov
Inference for population variances and proportions and intro to categorical data
**Reading:** Ch. 4.4, Ch. 6
.

Inference for the unknown variance $\sigma^2$ of a normal population

A marine biologist wishes to use male angelfish for an experiment and hopes their weights don't vary much. In fact, a previous random sample of $n = 16$ angelfish yielded the data below

$$\{y_1, \ldots, y_n\} = \{5.1, 2.5, 2.8, 3.4, 6.3, 3.6, 3.9, 3.0, 2.7, 5.7, 3.5, 3.6, 5.3, 5.1, 3.5, 3.3\}$$

Sample statistics from these data include

$$\bar{y} = 3.96 \text{ lbs} \quad s^2 = 1.35 \text{ lbs}^2 \quad n = 16$$

Problem: obtain a $100(1-\alpha)\%$ confidence interval for $\sigma^2$.

Point Estimator for $\sigma^2$? How about $S^2$?

Sampling theory for $S^2$?

If a random sample $Y_1, \ldots, Y_n$ is taken from a normal population with mean $\mu$ and variance $\sigma^2$, then

$$\frac{\sum(Y_j - \bar{Y})^2}{\sigma^2} \sim \chi^2(n-1)$$

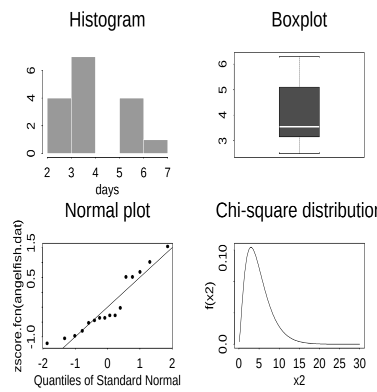
Critical values for the $\chi^2$ distribution appear in Table $C.3$ on pp 813-814 of Rao. These values cover distributions with up to $\nu = 100$ degrees of freedom. This result can be used to obtain confidence intervals for the variance $\sigma^2$ of a normal population:

$$1 - \alpha = \Pr\left(\chi^2(n-1, 1-\alpha/2) \le \frac{\sum(Y_j - \bar{Y})^2}{\sigma^2} \le \chi^2(n-1, \alpha/2)\right).$$

The term in the middle is just $(n-1)S^2/\sigma^2$. The usual algebraic rearrangement yields a confidence interval of the variance of the form

$$\boxed{\frac{(n-1)S^2}{\chi^2(n-1,\alpha/2)} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2(n-1,1-\alpha/2)}}.$$

Figure 1: Assessments of normality/sampling distribution of $(n-1)S^2/\sigma^2$:




For the angelfish data, first we might check for obvious departures from normality: To obtain a 95% confidence interval, the appropriate critical values are

$$\chi^2(15, 0.025) = 27.49 \quad \text{and } \chi^2(15, 0.975) = 6.26.$$

This yields the interval

$$\frac{(n-1)s^2}{\chi^2(n-1, \alpha/2)}, \; \frac{(n-1)s^2}{\chi^2(n-1, 1-\alpha/2)}$$

or

$$\frac{(16-1)1.35}{27.49}, \; \frac{(16-1)1.35}{6.26}$$

or

$$(0.74, 3.24)$$

The ratio of **two** population variances, $\sigma_1^2/\sigma_2^2$, from independent samples

Consider two independent random samples

$$Y_{1,1}, \ldots, Y_{1,n_1}$$

$$Y_{2,1}, \ldots, Y_{2,n_2}$$

from two **normal** populations with unknown variances $\sigma_1^2$ and $\sigma_2^2$, respectively. Questions:

- What is a good point estimator of $\sigma_1^2/\sigma_2^2$?
- Can this be used for a test of significance or confidence interval for $\sigma_1^2/\sigma_2^2$?

Sampling distributions of $S_1^2$ and $S_2^2$ from normal populations

Suppose we compare air pollution in homes of smokers and non-smokers. The common variances procedure was ruled out because of the large difference in sample variances:

$S_1^2 = 26.0$ ($n_1$ = 11) smokers
$S_2^2 = 195.4$ ($n_2$ = 9) non-smokers

Suppose we want to formally test the hypothesis that the population variances are equal. Consider a test of the form

$$H_0: \sigma_1^2 = \sigma_2^2 \text{ vs. } H_1: \sigma_1^2 \neq \sigma_2^2$$

which can also be written

$$H_0: \theta = \frac{\sigma_1^2}{\sigma_2^2} = 1 \text{ vs. } H_1: \theta = \frac{\sigma_1^2}{\sigma_2^2} \neq 1.$$

How about

$$\hat{\theta} = \frac{S_1^2}{S_2^2}?$$

Where
$S_1^2$ is the sample variance from $Y_{1,1}, \ldots, Y_{1,n_1}$ and
$S_2^2$ is the sample variance from $Y_{2,1}, \ldots, Y_{1,n_2}$:

$$S_1^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}(Y_{1,i} - \bar{Y}_1)^2$$
$$S_2^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}(Y_{2,i} - \bar{Y}_2)^2$$

($\hat{\theta}$ is sometimes called an $F$-ratio.)

To test $H_0$, the hypothesis of equality population variances, we use the following result:

$$\frac{\hat{\theta}}{\theta} \sim F_{n_1-1, n_2-1}$$

which can also be written

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1, n_2-1}.$$

This yields a probability statement of the form

$$1-\alpha = \Pr\left(F(n_1-1, n_2-1, 1-\alpha/2) \leq \frac{S_1^2}{S_2^2}\frac{1}{\theta} \leq F(n_1-1, n_2-1, \alpha/2) \;\middle|\; H_0 \text{ is true}\right) \tag{1}$$

Values of the F-ratio which are far from one constitute evidence against the null hypothesis. Formally, the critical region with level $\alpha$ calls for rejection of $H_0$ whenever

$$\hat{\theta} < F_{n_1-1, n_2-1}(1-\alpha/2) \qquad \text{or} \qquad \hat{\theta} > F_{n_1-1, n_2-1}(\alpha/2).$$

Manipulation of (1) leads to the following $100(1-\alpha)\%$ confidence interval for $\theta = \sigma_1^2/\sigma_2^2$:

$$\boxed{\left(\frac{S_1^2}{S_2^2}\frac{1}{F(n_1-1, n_2-1, \alpha/2)}, \quad \frac{S_1^2}{S_2^2}\frac{1}{F(n_1-1, n_2-1, 1-\alpha/2)}\right)}$$

For a 95% confidence interval for $\sigma_1^2/\sigma_2^2$ in the smoking data, we need

F(10; 8; 0.975) = 0.259 ; F(10; 8; 0.025) = 4.295

which yields the interval

$$\left(\frac{26.0}{195.4 \times 4.295}, \quad \frac{26.0}{195.4 \times 0.259}\right)$$

or

(0.031; 0.512)

which clearly doesn't contain 1, so that $H_0 : \sigma_1^2 = \sigma_2^2$ is rejected at level $\alpha = 0.05$.

The $p$-value for such a test can be obtained from the $F$ distribution and the observed test statistic:

$$F_{obs} = \hat{\theta}_{obs} = 26/195.4 = 0.133$$

However, recall that Table C.4 only gives upper critical values.

Therefore, to obtain a $p$-value, take as the test statistic

$$\max\{\hat{\theta}, 1/\hat{\theta}\}$$

and multiply the right-tail probability from the $F$-distribution by 2. Use the following numerator ($df_1$) and denominator ($df_2$) degrees of freedom:

$$\begin{aligned} df_1 &= df \text{ from bigger of } \{s_1^2, s_2^2\} \\ df_2 &= df \text{ from smaller of } \{s_1^2, s_2^2\} \end{aligned}$$

The observed test statistic becomes

$$F_{obs} = 1/\hat{\theta} = \frac{s_2^2}{s_1^2} = \frac{195.4}{26.0} = 7.53$$

and since

$$F(8, 10, 0.01) = 5.057$$

the area to the right of $F_{obs} = 7.53$ under the $F_{8,10}$ distribution is less than 0.01, which corresponds to a two-sided $p$-value less than 0.02. Note that the degrees of freedom must be switched when $S_2^2 > S_1^2$.

```
options ls=75 nodate;

data one;
   infile "datasets/smokers.dat";
   input y smoke;
   label y="suspended particulate matter";
run;

proc ttest;
   class smoke;
   var y;
run;
```

```
                              The SAS System                              1

                              TTEST PROCEDURE

Variable: Y            suspended particulate matter

SMOKE       N                 Mean              Std Dev             Std Error
-------------------------------------------------------------------------
    0       9          92.77777778          13.98014465            4.66004822
    1      11         133.18181818           5.09545252            1.53633674

Variances        T       DF    Prob>|T|
---------------------------------------
Unequal    -8.2343      9.7      0.0001
Equal      -8.9320     18.0      0.0000

For H0: Variances are equal, F' = 7.53    DF = (8,10)    Prob>F' = 0.0045
```

Large sample interval estimation for a population proportion

Out of a random sample of $n = 330$ triathletes, 167 indicated that they had suffered a training-related injury during the past year. Using these data, give a point estimate, standard error and confidence interval for

$p$ : the proportion among ALL triathletes who suffered an injury

Let

$$\hat{p} := \text{sample proportion of injured triathletes}$$

We know from the CLT for proportions that the sampling distribution of $\hat{p}$ is approximately normal. This yields the following approximate probability statement:

$$\begin{aligned}
0.95 &\approx \Pr\left(-1.96 < \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} < 1.96\right) \\
&\vdots \\
&= \Pr\left(\hat{p} - 1.96\sqrt{p(1-p)/n} < p < \hat{p} + 1.96\sqrt{p(1-p)/n}\right) \\
&\approx \Pr\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)
\end{aligned}$$

The endpoints for a 95% confidence interval for an unknown population proportion $p$ based on a random sample of size $n$ with sample proportion $\hat{p}$ are then given by

$$\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \text{ and } \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

which is commonly written

$$\boxed{\hat{p} \pm 1.96\sqrt{\tfrac{\hat{p}(1-\hat{p})}{n}}.}$$

For the triathlete data, a 95% confidence interval for $p$ based on the sample proportion of $\hat{p} = 167/330 = 0.506$ is given by

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

or

$$0.506 \pm 1.96(0.028)$$

or

$$0.506 \pm 0.053$$

Sample size computations for confidence intervals

Case 2: Estimation of a population proportion $p$.
The sample size necessary to obtain a 95% confidence interval of the form

$$\hat{p} \pm B$$

for an unknown population proportion $p$ based on a random sample can be solved for similarly, yielding the equation

$$n = \left( \frac{1.96\sqrt{\hat{p}(1-\hat{p})}}{B} \right)^2 . \qquad (2)$$

Upon inspection of (2), it can be seen that the term on the right is bounded above by

$$\left( \frac{1.96}{B} \right)^2 * (1/4)$$

so that a conservative sample size, which will ensure a 95% confidence interval of length 2*B is given by

$$n = \left( \frac{1.96}{B} \right)^2 * (1/4).$$

Exercise: Suppose you want to estimate the proportion $p$ of trees that will survive to a certain lifetime under some treatment of interest. In particular, you'd like a 95% confidence interval of the form

$$\hat{p} \pm 0.02.$$

How large does your sample size $n$ need to be ...

- without knowing anything about $p$?
- with the knowledge that the least $p$ could reasonably be is $p = 0.9$?

## Testing with dichotomous data

Example: There is a theory that the anticipation of a birthday can prolong a person's life. In a study, it was found that only $x = 60$ out of a random sample of $n = 747$ people whose obituaries were published in Salt Lake City in 1975 died in the three-month period preceding their birthday (Newsweek, 1978). Let $p$ denote the proportion of all deaths which fall in the three-month period preceding a birthday. Consider the following test

$$H_0 : p = 0.25 \ (= p_0) \quad \text{vs} \quad H_1 : p < 0.25$$

The test statistic for this problem takes the usual form

$$Z = \frac{\text{est} - \text{null}}{\text{SE(est)}} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/747}}$$

Note that the standard error term in the denominator does not need to be estimated (by $\hat{p}$) since it is specified under $H_0$. The left-tailed test with level $\alpha$ rejects $H_0$ if $Z < -z(\alpha)$. Similarly for right-tailed and two-tailed tests:

| Alternative | Critical region |
|---|---|
| $H_1 : p < p_0$ | $Z < -z(\alpha)$ |
| $H_1 : p > p_0$ | $Z > z(\alpha)$ |
| $H_1 : p \neq p_0$ | $\lvert Z \rvert > z(\alpha/2)$ |

For the Newsweek obituary data,

$$z_{obs} = \frac{60/747 - 0.25}{\sqrt{0.25(1-0.25)/747}} = \frac{0.08 - 0.25}{0.0099} = -17$$

So we reject $H_0$ with a $p$-value less than 0.001.

## Some "categorical" datasets:

Dataset #1: Tomato plants.

| Phenotype | Frequency |
|---|---|
| Tall, cut | 926 |
| Tall, potato | 288 |
| Dwarf, cut | 293 |
| Dwarf, potato | 104 |

Dataset #2: Yeast cells

The distribution of yeast cells observed over $n = 400$ squares of a haemacytometer:

| $y$ | 0 | 1 | 2 | 3 | 4 | 5 | $\geq 6$ |
|---|---|---|---|---|---|---|---|
| $f(y)$ | 213 | 128 | 37 | 18 | 3 | 1 | 0 |

Dataset #3: Colds among skiers taking vitamin C and placebo

| | Cold | No Cold | Total |
|---|---|---|---|
| Placebo | 31 | 109 | 140 |
| Vitamin C | 17 | 122 | 139 |

Dataset #4: Presidential candidates

| | | after debate | | |
|---|---|---|---|---|
| | | G | B | |
| before | G | 63 | 21 | 84 |
| debate | B | 4 | 12 | 16 |

Dataset #5: Handedness and gender

| Handedness | Men | Women | Total |
|---|---|---|---|
| Right | 934 | 1070 | 2004 |
| Left | 113 | 92 | 205 |
| Ambidextrous | 20 | 8 | 28 |
| Total | 1067 | 1170 | 2237 |

## The multinomial probability distribution

The **multinomial distribution** is a generalization of the binomial distribution arising from independent, identically distributed trials, each of which can be categorized as one and only one of $C \geq 2$ possible categories, with probabilities $\pi_1, \pi_2, \ldots, \pi_C$. If $n$ such i.i.d. trials are observed, each with probabilities $(\pi_1, \ldots, \pi_C)$ then the probability of obtaining exactly

- $y_1$ trials categorized as type 1
- $y_2$ trials categorized as type 2
- $\vdots$
- $y_C$ trials categorized as type C

is given by

$$\frac{n!}{y_1! y_2! \times \cdots \times y_C!} \pi_1^{y_1} \pi_2^{y_2} \times \cdots \times \pi_C^{y_C}$$

For example, if tomato plants are grown in such a way that they are classified as one of the four phenotypes in Dataset #1 with probabilities

$$\pi_1 = 0.56, \quad \pi_2 = 0.19, \quad \pi_3 = 0.19, \quad \pi_4 = 0.063$$

and $n = 10$ plants are grown, then the chance of getting, say exactly

| | |
|---|---|
| $y_1 = 5$ | Tall,cut |
| $y_2 = 2$ | Tall,potato |
| $y_3 = 2$ | Dwarf/cut |
| $y_4 = 1$ | Dwarf/potato |

is given by

$$\frac{10!}{5!2!2!1!} 0.56^5 0.19^2 0.19^2 0.063^1 = 0.033$$

Note: Results for the multinomial distribution underlie many of the techniques for categorical data analysis we'll study.

## The $\chi^2$ goodness-of-fit tests for categorical data with completely specified cell probabilities

The $\chi^2$ goodness-of-fit test can be used for inference about these $C-1$ parameters. (Since $\pi_1 + \pi_2 + \cdots + \ldots \pi_C = 1$ there are really only $C-1$ parameters.) In particular, it can be used to test hypotheses of the form

$$H_0 : \pi_1 = \pi_{10}, \pi_2 = \pi_{20}, \ldots, \pi_C = \pi_{C0}$$

versus

$$H_1 : \pi_j \neq \pi_{j0} \text{ for at least one } j$$

Suppose that $n$ i.i.d. trials are observed, each with probability of being classified (uniquely) as category $j$ given by $\pi_j$. Let the RV representing the number of trials classified as category $j$ be denoted by $O_j$:

$$O_j = \text{ \# trials classified as type} j.$$

Using properties of this multinomial distribution, it can be shown that when $H_0$ holds, the $\chi^2$ test statistic below has (approximately) the $\chi^2$ distribution with $C-1$ degrees of freedom:

$$\chi^2 = \sum_{j=1}^{j=C} \frac{(O_j - n\pi_{j0})^2}{n\pi_{j0}}$$

This test statistic is a bit easier to remember in the following form

$$\chi^2 = \sum_{j=1}^{j=C} \frac{(O_j - E_j)^2}{E_j}$$

where $O_j$ denotes the observed count in the $j^{th}$ category and $E_j$ is the expected count under $H_0$:

$$E_j = E(O_j; H_0) = n\pi_{j0}$$

A critical region for $\chi^2$ is the set of values bigger than $\chi^2(C-1, \alpha)$. That is,

$$\text{reject } H_0 \text{ if } \chi^2 \geq \chi^2(C-1, \alpha).$$

The $p$-value is just the area to the right of the observed value of the test statistic under the $\chi^2$ curve with $C-1$ degrees of freedom.

Example: Two traits that have been widely studied in tomato plants are *height* ("tall" vs "dwarf") and *leaf type* ("cut" vs "potato"). "Tall" and "cut" are dominant. When a homozygous "tall,cut" is crossed with a "dwarf,potato" the resulting progeny is called a dihybrid. When dihybrids are crossed, the following proportions of phenotypes should appear in the offspring provided the alleles governing the two traits segregate independently (this is a $9:3:3:1$ ratio:)

| Phenotype | Relative Frequency |
|---|---|
| Tall, cut | 0.5625 |
| Tall, potato | 0.1875 |
| Dwarf, cut | 0.1875 |
| Dwarf, potato | 0.0625 |

In one experiment done with these two traits a total of 1611 progeny of dihybrid crosses were categorized by phenotype. The data are summarized in the table below:

| Phenotype | Frequency |
|---|---|
| Tall, cut | 926 |
| Tall, potato | 288 |
| Dwarf, cut | 293 |
| Dwarf, potato | 104 |

Specify the null and alternative hypothesis for this experiment:

$$H_0 : ?$$

$$H_1 : ?$$

How about

$$H_0 : \pi_1 = 0.5625,\ \ \pi_2 = 0.1875,\ \ \pi_3 = 0.1875,\ \ \pi_4 = 0.0625$$

vs

$$H_1 : \text{at least one} \ \ \pi_j \neq \pi_{j0}?$$

To test these hypotheses, the $\chi^2$ test statistic becomes

$$\begin{aligned}
\chi^2 &= \sum_1^4 \frac{(O_j - E_j)^2}{E_j} \\
&= \frac{(926 - 906.2)^2}{906.2} + \frac{(288 - 302.1.2)^2}{302.1} + \frac{(293 - 302.1)^2}{302.1906} + \frac{(104 - 100.7)^2}{100.7} \\
&= 1.47
\end{aligned}$$

Is this statistically significant?

The distributional result on page 1 implies that when $H_0$ holds, the test statistic should have a $\chi^2$ sampling distribution with $4 - 1 = 3$ degrees of freedom. The $95^{th}$ percentile for this distribution, found in Rao, is given by

$$\chi^2(0.05, 3) = 7.8147$$

The observed test statistic is therefore not statistically significant using $\alpha = 0.05$. The $p$-value, obtained using statistical software is given by

$$p - \text{value} = \Pr(\chi^2 \geq 1.47; H_0) = 0.69.$$

Conclusion ?:

<u>Rule of thumb to check validity of $\chi^2$ approximation</u>

- at least 75% of the cells have $E_j \geq 5$ (Expected counts are not too small) AND
- no Expected 0's ( $E_j \neq 0$)

## Another test for categorical data: partially specified probabilities

Often, the category probability parameters are not completely specified, but rather are specified up to some unknown parameter. Examples include fitting a well-known discrete probability model, such as the poisson or binomial models to data, or making a continuous model into a discrete model by grouping observations in bins.

Example: a poisson probability model. The distribution of yeast cells observed over $n = 400$ squares of a haemacytometer is given below:

| $y$ | 0 | 1 | 2 | 3 | 4 | 5 | $\geq 6$ |
|---|---|---|---|---|---|---|---|
| $f(y)$ | 213 | 128 | 37 | 18 | 3 | 1 | 0 |

To test the hypothesis that these data are a random sample from a Poisson distribution, we could write

$$H_0 : \Pr(y \text{ yeast cells in a square}) = e^{-\lambda}\lambda^y/y! \text{ for } y = 0, 1, 2, \ldots$$

$$H_1 : \Pr(y \text{ yeast cells in a square}) \neq e^{-\lambda}\lambda^y/y! \text{ for } \text{ for some } y$$

however, there would be many zeroes and many small cell counts, so we can bin the data a bit differently to avoid this problem.

| Category $j$ | $j = 1$ | $j = 2$ | $j = 3$ | $j = 4$ | $j = 5$ |
|---|---|---|---|---|---|
| $y$ | 0 | 1 | 2 | 3 | $\geq 4$ |
| $f(y)$ | 213 | 128 | 37 | 18 | 4 |

Then we can test

$$\begin{aligned} H_0 : \quad & \pi_1 = e^{-\lambda}, \\ & \pi_2 = e^{-\lambda}\lambda, \\ & \pi_3 = e^{-\lambda}\lambda^2/2!, \\ & \pi_4 = e^{-\lambda}\lambda^3/3!, \\ & \pi_5 = 1 - \textstyle\sum_1^4 \pi_j \\ H_1 : \quad & \text{any other probabilities} \end{aligned}$$

only we don't know $\lambda$ and must estimate it from the data. This is what is meant by partially specified probabilities. The resulting test statistic below has an approximate $\chi^2$ distribution with $C - 1 - p$ degrees of freedom where $C$ denotes the number of categories or bins and $p$ denotes the number of parameters used to specify the category probabilities. For the poisson model, $p = 1$.

(For a normal model, where $\mu$ and $\sigma$ must be estimated by $\bar{Y}$ and $S$, the number of parameters would be $p = 2$.)

The mean of the sample, $\bar{Y}$ can be used to estimate $\lambda$, the mean of the poisson distribution:

$$\hat{\lambda} = \bar{y} = \frac{\sum y_j}{n} = \frac{273}{400} = 0.6825.$$

Substituting $\hat{\lambda}$ into the poisson model for the category probabilities yields the following expected cell counts:

| $y$ | 0 | 1 | 2 | 3 | $\geq 4$ |
|---|---|---|---|---|---|
| $j$ | 1 | 2 | 3 | 4 | 5 |
| $O_j$ | 213 | 128 | 37 | 18 | 4 |
| $\hat{\pi}_j$ | 0.505 | 0.345 | 0.118 | 0.028 | 0.005 |
| $E_j$ | 202.1 | 138.0 | 47.1 | 10.7 | 2.1 |

The $j = 3$ cell probability $\hat{\pi}_3$, for example, comes from

$$\hat{\pi}_3 = \frac{e^{-\hat{\lambda}}\hat{\lambda}^2}{2!} = 0.118$$

and the expected cell counts are just

$$E_j = n\hat{\pi}_j \text{ for } j = 1, \ldots, 5$$

The $\alpha = 0.05$ critical value for the $\chi^2$ test statistic can be obtained from the $\chi^2$ distribution with $C - 1 - p = 3$ degrees of freedom from Table C.3, (p. 814) of Rao:

$$\chi^2(3, 0.05) = 7.8147$$

The observed value of the test statistic is

$$\chi^2 = \sum \frac{(O_j - E_j)^2}{E_j} = \frac{(213 - 202.1)^2}{202.1} + \cdots + \frac{(4 - 2.1)^2}{2.1} = 10.12$$

Q: Does the test statistic fall in the $\alpha = 0.05$ critical region?

Q: Do the poisson model fit these data?

Q: What is the $p$-value for the test statistic $\chi^2$ under $H_0$?

$$\chi^2(3, 0.025) = 9.3 \text{ and } \chi^2(3, 0.01) = 11.3$$

## Large sample comparison of population proportions, $\pi_1, \pi_2$ based on independent random samples

Example: In a review of the evidence regarding the therapeutic value of vitamin C for prevention of the common cold, Pauling (1971) describes a 1961 French study involving 279 skiers during two periods of 5-7 days. One group of 140 subjects received a placebo while the remaining 139 received 1 gram of vitamin C per day. Of interest is the relative occurrence of colds for the two groups. The data are shown below. Let $p_1$ denote the proportion among a population of people who take the treatment who would catch a cold. Let $p_2$ denote the proportion among a population of people who take the placebo who would catch a cold.

| | Cold | No Cold | Total |
|---|---|---|---|
| Placebo | 31 | 109 | 140 |
| Vitamin C | 17 | 122 | 139 |

1. Formulate a test of hypotheses to investigate whether or not the catching of colds differs by vitamin C intake.
2. Calculate the $p-$value for your test from these data. If you use an approximation to obtain this $p-$value, verify that it is appropriate.
3. Obtain a 95% confidence interval for the quantity $p_1 - p_2$.
4. Let $q_1$ be defined by $q_1 = 1 - p_1$. Suppose that you are particularly interested in the quantity $\theta = q_1 - p_1$. Propose a point estimator of this quantity.
5. Construct a 95% confidence interval for $\theta$.

We've seen from the CLT for proportions that if $\hat{p}_1$ denotes a sample proportion (of some 0-1 trait of interest) from a random sample of size $n_1$ taken from a population with proportion $p_1$ then (approximately)

$$\frac{\hat{p}_1 - p_1}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}}} \sim N(0,1)$$

Similarly, if another sample proportion $\hat{p}_2$ is obtained from a random sample of size $n_2$ taken independently from another population with proportion $p_2$, then (approximately)

$$\frac{\hat{p}_2 - p_2}{\sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \sim N(0,1)$$

We also know that a sum or difference of two independent, normally distributed random variables also has a normal distribution. This implies that

$$\frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \sim N(0,1)$$

The following probability statement is a consequence of this normality:

$$1 - \alpha \approx \Pr\left(-z(\alpha/2) \le \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \le z(\alpha/2)\right)$$

The usual rearrangement yields a 95% confidence interval for $p_1 - p_2$ of the form

$$\boxed{\hat{p}_1 - \hat{p}_2 \pm z(\alpha/2)\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

For tests like $H_0 : p_1 - p_2 = D_0$ versus $H_1 : p_1 - p_2 \neq D_0$, the following test statistic can be used:

$$Z_1 = \frac{\hat{p}_1 - \hat{p}_2 - D_0}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

For the (most common) case where $D_0 = 0$ is of interest, a better test is one based on the statistic

$$Z_2 = \frac{\hat{p}_1 - \hat{p}_2 - D_0}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}}$$

where

$$\hat{p} = \frac{n_1}{n_1 + n_2}\hat{p}_1 + \frac{n_2}{n_1 + n_2}\hat{p}_2.$$

Critical regions for one-sided and two-sided alternatives are formed in the usual manner. It can be shown that $Z^2$ and the $\chi^2$ statistic for independence in a $2 \times 2$ table are the same (see pp. 20-21.)

For the vitamin C data, a 95% confidence interval for $p_p - p_C$ is given by $(0.004, 0.194)$. The 2nd test statistic works out to $z_{obs} = 2.19$ and a two-sided $p$-value of 0.0283:

```
data one;
   input cold trt $ frq;
   cards;
   1 p 31
   0 p 109
   1 C 17
   0 C 122
;
run;

proc freq;
   weight frq;
   tables cold*trt/chisq;
run;
```

```
                   The SAS System                              1
                TABLE OF TRT BY COLD

         TRT       COLD

         Frequency|
         Row Pct  |       0|       1|  Total
         ---------+--------+--------+
         C        |    122 |     17 |    139
                  |  87.77 |  12.23 |
         ---------+--------+--------+
         p        |    109 |     31 |    140
                  |  77.86 |  22.14 |
         ---------+--------+--------+
         Total         231       48      279

         STATISTICS FOR TABLE OF TRT BY COLD

Statistic                     DF     Value        Prob
------------------------------------------------------
Chi-Square                     1     4.811       0.028
Fisher's Exact Test (Left)                       0.991
                    (Right)                      0.021
                    (2-Tail)                     0.038
```

McNemar's test for significance of changes

McNemar's test can be used to test for a difference of proportions in paired categorical data. That is, two 0-1 measurements are made on each experimental unit. Consider hypothetical data representing preferences among democratic voters for a presidential candidate, $G$ or $B$, before and after a debate. Here, there are two measurements made on each experimental unit (democratic voters).

| | | after debate | | |
|---|---|---|---|---|
| | | G | B | |
| before | G | $a = 63$ | $b = 21$ | $N_1 = 84$ |
| debate | B | $c = 4$ | $d = 12$ | $N_2 = 16$ |
| Total | | $M_1 = 67$ | $M_2 = 33$ | $N = 100$ |

The difference in the proportion of people who support Gore before ($\pi_1$) and after ($\pi_2$) the debate, $\theta = \pi_1 - \pi_2$ can be estimated using

$$\hat{\theta} = \hat{\pi}_1 - \hat{\pi}_2 = \frac{N_1}{N} - \frac{M_1}{N}$$

For these data, this works out to

$$\hat{\theta} = \frac{84}{100} - \frac{67}{100} = \frac{63 + 21 - (63 + 4)}{100} = \frac{21 - 4}{100}$$

In general $(a, b, c, d)$ this estimator works out to

$$\hat{\theta} = \frac{b - c}{N}$$

It can be shown that the standard error can be estimated by of $\hat{\theta}$ is given by

$$SE(\hat{\theta}) = \frac{\sqrt{b + c}}{N}$$

yielding a test statistic for $H_0 : \pi_1 - \pi_2 = \theta_0$ of the form

$$Z = \frac{\hat{\theta} - \theta_0}{SE(\hat{\theta})}.$$

When $\theta_0 = 0$, this becomes

$$Z = \frac{b - c}{\sqrt{b + c}}.$$

In large samples, $Z \sim N(0, 1)$ and confidence intervals and tests can be constructed as usual.

For these hypothetical data, the test statistic becomes

$$Z_{obs} = \frac{21 - 4}{\sqrt{21 + 4}} = 3.4$$

which differs significantly from 0, indicating that Bush won the debate.

## $\chi^2$ test for independence

The $\chi^2$ test for independence can be used to detect independence among two categorical variables.

Example: A random sample of $n_{++} = 2237$ adults was conducted and there gender and handedness were observed and are tabulated below:

| Handedness | Men | Women | Total |
|---|---|---|---|
| Right | 934 | 1070 | 2004 |
| Left | 113 | 92 | 205 |
| Ambidextrous | 20 | 8 | 28 |
| Total | 1067 | 1170 | 2237 |

Define a RV $O_{ij}$ to model the observed counts for the cell in the $i^{th}$ row and $j^{th}$ column. Note the notational difference between rows and columns. Let the expected value for these RVs be denoted by $E_{ij}$ respectively.

| | Observed | | Expected | | |
|---|---|---|---|---|---|
| Handedness | Men | Women | Men | Women | Totals |
| Right | $O_{11}$ | $O_{12}$ | $E_{11}$ | $E_{12}$ | $n_{1+}$ |
| Left | $O_{21}$ | $O_{22}$ | $E_{21}$ | $E_{22}$ | $n_{2+}$ |
| Ambidextrous | $O_{31}$ | $O_{32}$ | $E_{31}$ | $E_{32}$ | $n_{3+}$ |
| Totals | $n_{+1}$ | $n_{+2}$ | | | $n_{++}$ |

Under the (null) hypotheses that handedness and gender are independent,

$$\Pr(\text{Left-handed} \cap \text{man}) = \Pr(\text{Left-handed}) \times \Pr(\text{man})$$

and so on for each gender and each category of handedness. So, an estimate for the number of left-handed men in the sample under this hypothesis is just the fraction of left-handers times the fraction of men times the sample size, or for general cell $(i, j)$:

$$E_{ij} = n_{++} \times \frac{n_{i+}}{n_{++}} \times \frac{n_{+j}}{n_{++}} = n_{++} \times n_{i+} \times n_{+j}$$

| | Observed | | Expected | |
|---|---|---|---|---|
| Handedness | Men | Women | Men | Women |
| Right | 934 | 1070 | 955.9 | 1048.1 |
| Left | 113 | 92 | 97.8 | 107.2 |
| Ambidextrous | 20 | 8 | 13.4 | 14.6 |

Then the $\chi^2$ test for independence in an $I \times J$ contingency table is based upon the test stastistic

$$\chi^2 = \sum_{i=1}^{I}\sum_{j=1}^{J} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

which has a $\chi^2$ distribution with degrees of freedom given by $(I - 1) \times (J - 1)$ under the null hypothesis of independence. In our example,

$$\chi^2 = \left[\frac{(934 - 955.9)^2}{955.9} + \frac{(1070 - 1048.1)^2}{1048.1} + \cdots + \frac{(8 - 14.6)^2}{14.6}\right] \approx 12$$

The critical value for this test statistic is $\chi^2(2, 0.05) = 5.99$.

$H_0$ and $H_1$?

Conclusion: ?

$p$-value: ?