Chapter 2

Probability

2.1 Sample Spaces and Events

The sample space of an experiment, denoted \( S \), is the set of all possible outcomes of that experiment.

Sample Space

Ex. Roll a die. Outcomes: landing with a 1, 2, 3, 4, 5, or 6 face up.

Sample Space: \( S = \{1, 2, 3, 4, 5, 6\} \)

Events

An event is any collection (subset) of outcomes contained in the sample space \( S \). An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcome.
Relations from Set Theory

1. The **union** of two events $A$ and $B$ is the event consisting of all outcomes that are either in $A$ or in $B$.
   
   Notation: $A \cup B$
   Read: $A$ or $B$

2. The **intersection** of two events $A$ and $B$ is the event consisting of all outcomes that are in both $A$ and $B$.
   
   Notation: $A \cap B$
   Read: $A$ and $B$

3. The **complement** of an event $A$ is the set of all outcomes in $S$ that are not contained in $A$.
   
   Notation: $A'$

Events

**Ex.** Rolling a die. $S = \{1, 2, 3, 4, 5, 6\}$
   
   Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$
   
   $A \cup B = \{1, 2, 3, 5\}$
   $A \cap B = \{1, 3\}$
   $A' = \{4, 5, 6\}$

Mutually Exclusive

When $A$ and $B$ have no outcomes in common, they are **mutually exclusive** or **disjoint** events

**Ex.** When rolling a die, if event $A = \{2, 4, 6\}$ (evens) and event $B = \{1, 3, 5\}$ (odds), then $A$ and $B$ are mutually exclusive.

**Ex.** When drawing a single card from a standard deck of cards, if event $A = \{heart, diamond\}$ (red) and event $B = \{spade, club\}$ (black), then $A$ and $B$ are mutually exclusive.
2.2
Axioms, Interpretations, and Properties of Probability

Axioms of Probability

- **Axiom 1** \( P(A) \geq 0 \) for any event \( A \)
- **Axiom 2** \( P(S) = 1 \)
- If all \( A_i \)'s are mutually exclusive, then
  
  **Axiom 3** \( P(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{i=1}^{k} P(A_i) \) (finite set)

\[
P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)
\] (infinite set)

Properties of Probability

- For any event \( A \), \( P(A) = 1 - P(A^c) \).
- If \( A \) and \( B \) are mutually exclusive, then
  \( P(A \cap B) = 0 \).
- For any two events \( A \) and \( B \),
  
  \[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
  \]

Ex. A card is drawn from a well-shuffled deck of 52 playing cards. What is the probability that it is a queen or a heart?

\( Q = \) Queen and \( H = \) Heart

\[
P(Q) = \frac{4}{52}, \quad P(H) = \frac{13}{52}, \quad P(Q \cap H) = \frac{1}{52}
\]

\[
P(Q \cup H) = P(Q) + P(H) - P(Q \cap H)
\]

\[
= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}
\]

2.3
Counting Techniques
**Product Rule**

If the first element or object of an ordered pair can be used in \( n_1 \) ways, and for each of these \( n_1 \) ways the second can be selected \( n_2 \) ways, then the number of pairs is \( n_1n_2 \).

**Note that this generalizes to \( k \) elements (\( k \)-tuples)**

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**Permutations**

Any ordered sequence of \( k \) objects taken from a set of \( n \) distinct objects is called a **permutation** of size \( k \) of the objects.

Notation: \( P_{k,n} \)

\[
P_{k,n} = n(n-1)\cdots(n-k+1)
\]

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**Factorial**

For any positive integer \( m \), \( m! \) is read “\( m \) factorial” and is defined by 

\[
m! = m(m-1)\cdots(2)(1).
\]

Also, \( 0! = 1 \).

Note, now we can write:

\[
P_{k,n} = \frac{n!}{(n-k)!}
\]

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**Combinations**

Given a set of \( n \) distinct objects, any unordered subset of size \( k \) of the objects is called a **combination**.

Notation: \( \binom{n}{k} \) or \( C_{k,n} \)

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

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**Ex.** A boy has 4 beads – red, white, blue, and yellow. How many different ways can three of the beads be strung together in a row?

This is a permutation since the beads will be in a row (order).

\[
P_{3,4} = \frac{4!}{(4-3)!} = 4! = 24
\]

24 different ways

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**Ex.** A boy has 4 beads – red, white, blue, and yellow. How many different ways can three of the beads be chosen to trade away?

This is a combination since they are chosen without regard to order.

\[
\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4
\]

4 different ways
Ex. Three balls are selected at random without replacement from the jar below. Find the probability that one ball is red and two are black.

\[
\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} = \frac{2 \cdot 3}{56} = \frac{3}{28}
\]

2.4 Conditional Probability

Conditional Probability

For any two events \(A\) and \(B\) with \(P(B) > 0\), the **conditional probability** of \(A\) given that \(B\) has occurred is defined by

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}
\]

Which can be written:

\[
P(A \cap B) = P(B) \cdot P(A \mid B)
\]

The Law of Total Probability

If the events \(A_1, A_2, \ldots, A_k\) be mutually exclusive and exhaustive events. The for any other event \(B\),

\[
P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)
\]

Ex. A store stocks light bulbs from three suppliers. Suppliers \(A\), \(B\), and \(C\) supply 10%, 20%, and 70% of the bulbs respectively. It has been determined that company \(A\)'s bulbs are 1% defective while company \(B\)'s are 3% defective and company \(C\)'s are 4% defective. If a bulb is selected at random and found to be defective, what is the probability that it came from supplier \(B\)?

\[
P(D) = \frac{P(D \mid A)P(A) + P(D \mid B)P(B) + P(D \mid C)P(C)}{0.2(0.03) + 0.1(0.01) + 0.7(0.04)} = 0.1714
\]

So about 0.17
2.5 Independence

Independent Events

Two events $A$ and $B$ are independent events if $P(A \mid B) = P(A)$.

Otherwise $A$ and $B$ are dependent.

**Note:** this generalizes to more than two independent events.

Permutation & Combination

**Permutation:** Number of ordered arrangements of $r$ objects chosen from $n$ distinctive objects

$$P_n^r = n(n-1)(n-2)\ldots(n-r+1)$$

$$P_n^r = P_n^{n-r} \cdot P_r^r$$

E.g., $P_6^3 = 6 \cdot 5 \cdot 4 = 120$.

**Combination:** Number of non-ordered arrangements of $r$ objects chosen from $n$ distinctive objects:

$$C_n^r = \frac{P_n^r}{P_r^r} = \frac{n!}{(n-r)!r!}$$

Or use notation of:

E.g., $5! = 120$, $0! = 1$

$$C_7^3 = \frac{7!}{4!3!} = 35$$

Theory of Counting = Combinatorial Analysis

Generalized Principle of Counting: If $M$ (independent) experiments are performed and the first one has $N_m$ possible outcomes, $1 \leq m \leq M$, then the total number of outcomes of the combined experiment is

$$N_1 \times N_2 \times \ldots \times N_M$$

E.g., How many binary functions $f(i) = 0$ or $f(i) = 1$, defined on a grid $1, 2, 3, \ldots, n$, are there? How many numbers can be stored in 8 bits = 1 byte?

$$2 \times 2 \times \ldots \times 2 = 2^n$$
Combinatorial Identity:
\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}
\]

Analytic proof: (expand both hand sides)
Combinatorial argument: Given \(n\) objects focus on one of them (obj. 1). There are \(n-1\) groups of size \(r\) that contain obj. 1 (since each group contains \(r-1\) other elements out of \(n-1\)). Also, there are \(n-1\) groups of size \(r\), that do not contain obj1. But the total of all \(r\)-size groups of \(n\)-objects is \(\binom{n}{r}\).

Examples
1. Suppose car plates are 7-digit, like AB1234. If all the letters can be used in the first 2 places, and all numbers can be used in the last 4, how many different plates can be made? How many plates are there with no repeating digits?
Solution: a) \(26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10\)

b) \(P_{26}^2 \cdot P_{10}^4 = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7\)

Examples
2. How many different letter arrangement can be made from the 11 letters of MISSISSIPPI?
Solution: There are: 1 M, 4 I, 4 S, 2 P letters.
Method 1: consider different permutations:
\[
\frac{11!}{1!4!4!2!} = 34650
\]
Method 2: consider combinations:
\[
\binom{11}{1} \binom{10}{4} \binom{6}{2} = \ldots = \binom{11}{2} \binom{9}{4} \binom{5}{1} \binom{1}{1}
\]

Examples
3. There are \(N\) telephones, and any 2 phones are connected by 1 line. Then how many lines are needed all together?
Solution: \(C^2_N = \frac{N \cdot (N - 1)}{2}\)
If, \(N=5\), complete graph with 5 nodes has \(C^2_5=10\) edges.

Examples
4. \(N\) distinct balls with \(M\) of them white. Randomly choose \(n\) of the \(N\) balls. What is the probability that the sample contains exactly \(m\) white balls (suppose every ball is equally likely to be selected)?
Solution: a) For the event to occur, \(m\) out of \(M\) white balls are chosen, and \(n-m\) out of \(N-M\) non-white balls are chosen. And we get
\[
\binom{M}{m} \binom{N-M}{n-m}
\]

b) Then the probability is
\[
\binom{M}{m} \binom{N-M}{n-m} / \binom{N}{n}
\]
Later these probabilities will be associated with the name HyperGeometric \((N, n, M)\) distribution.
Examples

5. N boys (♂) and M girls (♀), M<=N+1, stand in 1 line. How many arrangements are there so that no 2 girls stand next to each other?

Solution: $N! \cdot \frac{N+1}{M!}$

There are $M!$ ways of ordering the girls among themselves. NOTE: if girls are indistinguishable then there’s no need for this factor!

There are $N!$ ways of ordering the boys among themselves.

There are $N+1$ slots for the girls to fill between the boys. And there are $M$ girls to position in these slots, hence the coefficient in the middle.

How about they are arranged in a circle?

Answer: $N! \cdot \frac{M}{M!}$

E.g., N=3, M=2

Examples

5a. How would this change if there are N functional (♂) and M defective chips (♀), M<=N+1, in an assembly line?

Solution: $\binom{N+1}{M}$

There are $N+1$ slots for the girls to fill between the boys. And there are $M$ girls to position in these slots, hence the coefficient in the middle.

Binomial theorem & multinomial theorem

Binomial theorem $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Generalization: Divide n distinctive objects into k groups, with the size of every group $n_1, \ldots, n_k$ and $n_1 + n_2 + \ldots + n_k = n$

$$\binom{v_1, \ldots, v_k}{n_1, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Multinomial Probabilities

$$P(n_1, \ldots, n_k) = \frac{n!}{n_1! \cdots n_k!} \cdot p_1^{n_1} \cdots p_k^{n_k}$$

Multinomial theorem – will discuss in Ch. 03
Examples

7. There are n balls randomly positioned in r distinguishable urns. Assume n ≥ r. What is the number of possible combinations?

1) If the balls are distinguishable (labeled): \( r^n \) possible outcomes, where empty urns are permitted. Since each of the \( n \) balls can be placed in any of the \( r \) urns.

2) If the balls are indistinguishable: no empty urns are allowed — select \( r-1 \) of all possible \( n-1 \) dividing points between the \( n \)-balls.

3) If empty urns are allowed:

\[
\binom{n+r-1}{r-1} = \binom{n+r-1}{n}
\]

n=9, r=3 and two bins are empty

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Application – Number of integer solutions to linear equation

1) There are \( \binom{n+r-1}{r-1} \) distinct positive integer-valued vectors \( (x_1, x_2, \ldots, x_r) \) satisfying

\[
x_1 + x_2 + \ldots + x_r = n, \quad x_i > 0, \quad 1 \leq i \leq r
\]

2) There are \( \binom{n+r-1}{r-1} \) distinct positive integer-valued vectors \( (y_1, y_2, \ldots, y_r) \) satisfying

\[
y_1 + y_2 + \ldots + y_r = n, \quad x_i \geq 0, \quad 1 \leq i \leq r
\]

Since there are \( n+r-1 \) possible positions for the dividing splitters (or by letting \( y_i = x_i - 1 \), RHS = \( n+r \)).

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Examples

8. Randomly give \( n \) pairs of distinctive shoes to \( n \) people, with 2 shoes to everyone. How many arrangements can be made? How many arrangements are there, so that everyone gets an original pair? What is the probability of the latter event, \( \Pr(E) \)?

Solution: a) according to

\[
\binom{2n}{r} = \binom{2n}{n}
\]

Note: \( r = n = \# \text{ of pairs!} \)

total arrangements is

\[
N = (2n)!/(2!)^r = (2n)!/(2n!)
\]

b) Regard every shoe pair as one object, and give them to people, there are \( M = n! \) arrangements.

\[
\Pr(E) = 1/M \times n! / [(2n)!/2!] = (1/2)^n
\]

(Do \( n=6 \), case by hand!)

*note: \( n! = (n-2)(n-4) \ldots \)

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Sterling Formula for asymptotic behavior of \( n! \)

Sterling formula:

\[
n! = \sqrt{2\pi} \times \left( \frac{n}{e} \right)^n
\]

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Probability and Venn diagrams

Venn’s diagram

Union: \( A \cup B \)
Intersection: \( A \cap B \)
\( A \) denotes the part in \( \Omega \) but not in \( A \).

Properties:

\[
\begin{align*}
(A \cup B)' &= A' \cap B', \\
(A \cap B)' &= A' \cup B', \\
(A \cup B) \cap (A \cap B) &= A, \\
(A \cap B) \cup (A \cup B) &= \Omega, \\
(A \cup B)' &= A' \cap B'
\end{align*}
\]

De Morgan’s Laws:

\[
\begin{align*}
(A \cup B)' &= A \cap B', \\
(A \cup B)' &= (A' \cup B' \cap (A' \cap B'))
\end{align*}
\]

Generalized: \( \bigcup_{i=1}^{n} E_i \cap \bigcap_{i=1}^{n} F_i \cap \bigcup_{i=1}^{n} G_i \cap \bigcap_{i=1}^{n} H_i \)
### Probability and Venn diagrams

**Proposition**

\[
P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \ldots
\]

\[
+ (-1)^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-1} \leq n} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_{n-1}}) + \ldots
\]

\[
+ (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)
\]

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### Let's Make a Deal Paradox – aka, Monty Hall 3-door problem

- This paradox is related to a popular television show in the 1970’s. In the show, a contestant was given a choice of three doors/cards of which one contained a prize (diamond). The other two doors contained gag gifts like a chicken or a donkey (clubs).

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### Let's Make a Deal Paradox.

- After the contestant chose an initial door, the host of the show then revealed an empty door among the two unchosen doors, and asks the contestant if he or she would like to switch to the other unchosen door. The question is should the contestant switch. Do the odds of winning increase by switching to the remaining door?

- 1. Pick one card
- 2. Show one Club Card
- 3. Change 1st pick?

- The intuition of most people tells them that each of the doors, the chosen door and the unchosen door, are equally likely to contain the prize so that there is a 50-50 chance of winning with either selection? This, however, is not the case.

- The probability of winning by using the switching technique is 2/3, while the odds of winning by not switching is 1/3. The easiest way to explain this is as follows: