

# STAT 35A HW3 Solutions

[http://www.stat.ucla.edu/~dinov/courses\\_students.dir/09/Spring/STAT35.dir](http://www.stat.ucla.edu/~dinov/courses_students.dir/09/Spring/STAT35.dir)

1. Shortly after being put into service, some buses manufactured by a certain company have developed cracks on the underside of the main frame. Suppose a particular city has 20 of these buses, and cracks have actually appeared in 8 of them.

1.A: HOW MANY WAYS ARE THERE TO SELECT A SAMPLE OF 5 BUSES FROM THE 20 FOR A THOROUGH INSPECTION?

The order does not matter, so we want to use the combination:  $\binom{20}{5} = \frac{20!}{5! \cdot 15!} = 15504$

1.B: IN HOW MANY WAYS CAN A SAMPLE OF 5 BUSES CONTAIN EXACTLY 4 WITH VISIBLE CRACKS?

Of the 8 busses that have cracks, we want to select 4 of them. The remaining bus must not have a crack, for which

there are 12. The total number of arrangements of 5 busses that have 4 cracks is then  $\binom{8}{4} \cdot \binom{12}{1} = 840$ .

1.C: IF A SAMPLE OF 5 BUSES IS CHOSEN AT RANDOM, WHAT IS THE PROBABILITY THAT EXACTLY 4 OF THE 5 WILL HAVE VISIBLE CRACKS?

Dividing part 1.b with 1.a, we get the probability of getting exactly 4 cracks from 5 busses as  $\frac{\binom{8}{4} \cdot \binom{12}{1}}{\binom{20}{5}} = .05417$ . Just like the ball-and-urn problem without replacement, this event follows the hypergeometric distribution.

To further illustrate this, let's compute the counts for all possible numbers of cracks:

Cracks	Combinations
0	$\binom{12}{5} = 792$
1	$\binom{8}{1} \binom{12}{4} = 3960$
2	$\binom{8}{2} \binom{12}{3} = 6160$
3	$\binom{8}{3} \binom{12}{2} = 3696$
4	$\binom{8}{4} \binom{12}{1} = 840$

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5	$\binom{8}{5} = 56$
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Summing all these up, we get  $792+3960+6160+3696+840+56=15504$ , which is precisely the total number of ways to draw 5 busses, or  $\binom{20}{5}$ .

1.D: IF BUSES ARE SELECTED AS IN PART (C), WHAT IS THE PROBABILITY THAT AT LEAST 4 OF THOSE SELECTED WILL HAVE VISIBLE CRACKS?

Using the table computed above, simply sum the counts for 4 and 5 cracks, and divide the result by the total number of possibilities:  $\frac{840 + 56}{\binom{20}{5}} = .05779$ .

2. Three molecules of type A, 3 of type B, 3 of type C, and 3 of type D are to be linked together to form a chain molecule. One such chain molecule is ABCDABCDABCD, and another is BCDDAAABDBCC.

2.A: HOW MANY SUCH CHAIN MOLECULES ARE THERE?

If all 12 components of the chain were unique, then we would have 12! different orderings. Because this chain is composed of 4 groups of 3 identical models, there are 3! different orderings for each triple. The total number of

chains is then  $\frac{12!}{(3!)^4} = 369600$ .

2.B: SUPPOSE A CHAIN MOLECULE OF THE TYPE DESIRED IS RANDOMLY SELECTED. WHAT IS THE PROBABILITY THAT ALL THREE MOLECULES OF EACH TYPE END UP NEXT TO ONE ANOTHER (SUCH AS BBBAADDCCC)?

If all three molecules must appear next to each other, we can represent them as a single entity (A' = AAA, B'=BBB, etc). The total number of arrangements of these 4 entities is simply 4!. We want the probability of this event,

which we get by dividing by the total number of possible chains:  $\frac{4!}{369600} = .000065$

3. A certain sports car comes equipped with either an automatic or a manual transmission, and the car is available in one of four colors. Relevant probabilities for various combinations of transmission type and color are given in the table below. Let  $A = \{\text{automatic transmission}\}$ ,  $B = \{\text{black}\}$ , and  $C = \{\text{white}\}$ .

		Color			
		White	Blue	Black	Red
Transmission type	A	.15	.10	.10	.10
	M	.15	.05	.15	.20

3.A: CALCULATE  $P(A)$ ,  $P(B)$ , AND  $P(A \cap B)$ .

$$P(A) = .15 + .1 + .1 + .1 = .45$$

$$P(B) = .1 + .15 = .25$$

$$P(A \cap B) = .1$$

3.B: CALCULATE BOTH  $P(A | B)$  AND  $P(B | A)$ , AND EXPLAIN IN CONTEXT WHAT EACH OF THESE PROBABILITIES REPRESENT.

$$P(A | B) = P(A \cap B) / P(B) = .1 / .25 = .4$$

Given that we know the color is black, we want to know the probability the transmission is automatic. This is just the joint event of  $P(A \cap B)$  renormalized as a proportion of  $P(B)$ .

3.C: CALCULATE AND INTERPRET  $P(A | C)$  AND  $P(A | C')$ .

$$P(A | C) = P(A \cap C) / P(C) = .15 / .3 = .5$$

$$P(A | C') = P(A \cap C') / P(C') = (.1 + .1 + .1) / (.1 + .1 + .1 + .05 + .15 + .20) = .4286$$

**4. Use the Die Coin Experiments, from SOCR Experiments, to empirically validate the Bayesian rule for computing conditional probabilities:**

4.A: SUPPOSE D3 = EVENT { DIE TURNS UP A THREE (FIRST PART OF THE EXPERIMENT) } AND 2H = EVENT { THERE ARE TWO HEADS OBSERVED IN THE TOSSED COINS (SECOND PART OF THE EXPERIMENT) }. RUN THE EXPERIMENT 100 TIMES FIRST WITH FLAT-DIE-PROBABILITIES ( $P=1/6$ ) AND A FAIR-COIN ( $P=0.5$ ). BY COUNTING OUTCOMES OF INTEREST VALIDATE COMPUTATIONALLY THAT:

$$O \ P(D3 | 2H) = [ P(2H | D3) P(D3) ] / P(2H)$$

In my 100 experiments, 2H occurred 32 times, D3 occurred 15 times. D3 and 2H occurred together 7 times. Therefore:

$$P(D3) = .15$$

$$P(2H) = .32$$

$$P(D3 \cap 2H) = .07$$

$$P(D3 | 2H) = 7 / 32 = \mathbf{.21875}$$

$$P(2H | D3) = 7 / 15 = .46667$$

$$(P(2H | D3)P(D3)) / P(2H) = (.46667 * .15) / .32 = \mathbf{.21875}$$

4.B: NOW RUN THE EXPERIMENT 100 TIMES FIRST WITH A LOADED DIE WITH PROBABILITIES ( $P_1= 0.01$ ,  $P_2=0.05$ ,  $P_3=0.1$ ,  $P_4=0.2$ ,  $P_5=0.34$ ,  $P_6=0.3$ ) AND A LOADED COIN ( $P=0.3$ ). AGAIN BY COUNTING OUTCOMES OF INTEREST VALIDATE COMPUTATIONALLY THAT:

$$O \ P(D3 | 2H) = [ P(2H | D3) P(D3) ] / P(2H)$$

2H occurred 26 times, D3 occurred 14 times, D3 and 2H occurred together 4 times.

$$P(D3) = .14$$

$$P(2H) = .26$$

$$P(D3 \cap 2H) = .04$$

$$P(D3 | 2H) = 4 / 26 = \mathbf{.15385}$$

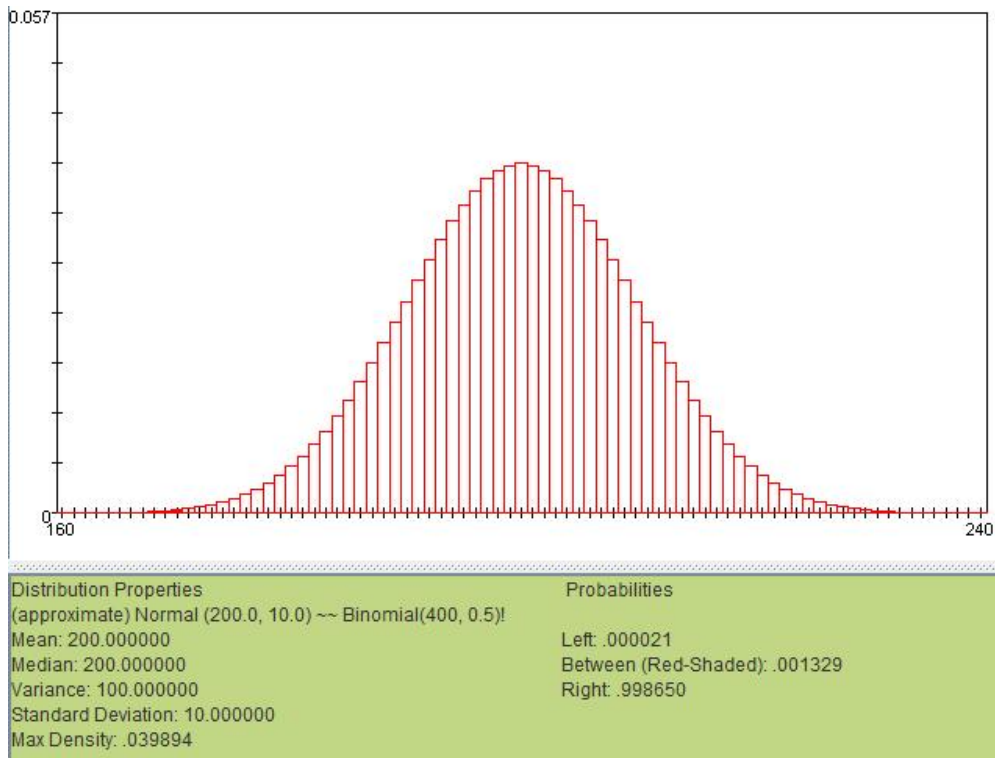
$$P(2H | D3) = 4 / 14 = .28571$$

$$(P(2H | D3)P(D3)) / P(2H) = (.28571 * .14) / .26 = \mathbf{.15385}$$

5. A coin is tossed 400 times and 170 heads are observed. This coin is (choose one answer):

- (a) fair, because the probability of seeing that amount of heads or less is approximately 0.0013
- (b) neither fair or unfair. There is not enough information to determine that.
- (c) fair, because the probability of seeing that amount of heads or less is approximately 0.5
- (d) not fair, because the probability of seeing that amount of heads or less is close to 0.

Let's first assume the coin is fair, we can validate the probability of observing less than 170 heads is indeed .0013 using the SOCR distribution tool by summing the area left of 170. This is quite low, as we would only expect to see this event occur once in 750 trials with a fair coin. We can conclude that the coin is most likely not fair, but we can't entirely rule it out because the probability isn't so close to zero as to be impossible. Therefore we would probably need additional trials to gain more confidence in our conclusion.



6. Two cards are dealt to you (without replacement) from an ordinary well-shuffled deck. Let  $X$  = the probability that you have a pair. Let  $Y$  = the probability that both of your cards are diamonds. Compare  $X$  and  $Y$ . Choose one answer.

(a)  $X < Y$

(b)  $X = Y$

(c)  $X > Y$

The probability of getting two diamonds is  $\frac{\binom{13}{2}}{\binom{52}{2}} = \frac{78}{1326} = .05882$

The probability of getting a pair is  $\frac{\binom{13}{1}\binom{4}{2}}{\binom{52}{2}} = \frac{78}{1326} = .05882$

These events have the same probability, so (b)  $X=Y$ .

7. Design an algorithm that takes a pair integers ( $n \geq k$ ) as input, generates explicitly all possible permutations of  $n$  elements (the total number of labeled objects) taken  $k$  at a time ( $k$  is the size of the sample we form from the available  $n$  objects), print the total number of these arrangements and print a list of all of these permutations. It's recommended that you implement a computer program (in any language) to test and validate your algorithm. At a minimum, if you only include the algorithm, but did not implement and test a computer program, then manually walk through the algorithm and validate that it produces the right number or permutations for  $n \leq 4$ .

There are many different ways to solve this problem. I chose to write a small computer program in the c# language. My program has two subroutines: one that finds all orderings of a list of  $k$  symbols, and one that finds all possible subsets of  $k$  symbols from  $n$  symbols. Both subroutines compute their results recursively. The entire program is listed below:

```
class Program {
    private static int _totalCount = 0;

    // Print out all k! orderings of the symbol list s
    public static void EnumerateOrderings(char[] s, int index) {
        if (index == s.Length) {
            Console.WriteLine(s);
            _totalCount++;
        }
        for (int i = index; i < s.Length; i++) {
            char tmp = s[index];
            // Swap elements at index and i
            s[index] = s[i];
            s[i] = tmp;
            EnumerateOrderings(s, index + 1);
            // Swap them back
            s[i] = s[index];
            s[index] = tmp;
        }
    }

    // Find all possible subsets of length k, from n total symbols
    public static void Permute(char[] s, char[] subset, int index, int depth) {
        if (depth == 0) {
            EnumerateOrderings(subset, 0);
            return;
        }
        for (int i = index; i <= s.Length - depth; i++) {
            subset[subset.Length - depth] = s[i];
            Permute(s, subset, i + 1, depth - 1);
        }
    }

    // Main function. Extract a list of symbols of length n,
    // and the value for k from the command line
    public static void Main(string[] args) {
        if (args.Length < 2) {
            return;
        }
        char[] s = args[0].ToCharArray();
        int k = Int32.Parse(args[1]);
        char[] subset = new char[k];

        // Print out all permutations
        Permute(s, subset, 0, k);
    }
}
```

```
        Console.WriteLine("{0} total permutations", _totalCount);  
    }  
}
```

Executing the program with the symbol list {a, b, c, d, e} and k=3, I get the following results, which is consistent with the permutation calculation  $5!/(5-3)! = 120/2 = 60$ :

```
C:\Dev\Permutation\bin\debug\Permutation.exe abcde 3
```

```
abc
acb
bac
bca
cba
cab
abd
adb
bad
bda
dba
dab
abe
aeb
bae
bea
eba
eab
acd
adc
cad
cda
dca
dac
ace
aec
cae
cea
eca
eac
ade
aed
dae
dea
eda
ead
bcd
bdc
cbd
cdb
dcb
dbc
bce
bec
cbe
ceb
ecb
ebc
bde
bed
dbe
deb
edb
ebd
cde
ced
dce
dec
edc
ecd
60 total permutations
```