Homework 03 Solution

Stats 13, Section 1, Spring 2013

1. Suppose a family has 2 children. Let D be the event that an offspring has the disease. Further, let M be the event that the offspring is male. Then

   - $P(M) = 0.51$ and $P(M^c) = 0.49$,
   - $P(D|M) = 0.5$ and $P(D^c|M) = 0.5$,
   - $P(D|M^c) = 0.0$ and $P(D^c|M^c) = 1.0$.
   - By the law of total probability (Hint: Draw a tree diagram if you want to visually see how to do the following calculation),

   $$\begin{aligned} P(D) &= P(D|M) \cdot P(M) + P(D|M^c) \cdot P(M^c) \\ &= 0.5 \cdot 0.51 + 0.0 \cdot 0.49 \\ &= 0.255 \end{aligned}$$

     which implies
   $$P(D^c) = 1 - P(D) = 0.745.$$

   (a) What is the probability that both sibling are affected, if there are one male and one female?

   $$\begin{aligned} P\left[(D|M) \cap (D|M^c)\right] &= P(D|M) \cdot P(D|M^c) \\ &= 0.5 \cdot 0 \\ &= 0 \end{aligned}$$

   (b) What is the probability that exactly one sibling is affected?

   Let X : Number of children that have the disease. Then

   $$X \sim Bin(2, 0.255)$$

   and
   $$P(X = 1) = \binom{2}{1}(0.255)^1(0.745)^1 = 0.37995.$$

   (c) What is the probability that neither sibling is affected?

   $$P(X = 0) = \binom{2}{0}(0.255)^0(0.745)^2 = 0.555.$$

2. Let $+$ be the event that a person tests positive for the disease and let D be the event that a person is diseased. Then

- $P(+|D) = 0.73$ and $P(+^c|D) = 0.27$.
- $P(+|D^c) = 0.11$ and $P(+^c|D^c) = 0.89$.
- $P(D) = 0.15$ and $P(D^c) = 0.85$.

(a) What is the probability that a randomly chosen person will test positive?

By the law of total probability (Hint: Draw a tree diagram if you want to visually see how to do the following calculation),

$$\begin{aligned} P(+) &= P(+|D)P(D) + P(+|D^c)P(D^c) \\ &= 0.73 \cdot 0.15 + 0.11 \cdot 0.86 \\ &= 0.2041 \end{aligned}$$

(b) Suppose that a randomly chosen person does test positive. What is the probability that the person does have the disease?

By Baye's rule (Hint: Draw a tree diagram if you want to visually see how to do the following calculation),

$$\begin{aligned} P(D|+) &= \frac{P(+|D)P(D)}{P(+)} \\ &= \frac{0.73 \cdot 0.15}{0.2041} \\ &= 0.5365 \end{aligned}$$

3. We start with the following table for 5,000 European Starlings. Let Y : Size of the brood.

Brood Size	Brood Number
1	90
2	230
3	610
4	1300
5	1810
6	800
7	130
8	26
9	3
10	1
Total	5000

(a)

$$P(Y = 4) = \frac{1300}{5000} = 0.26.$$

(b)

$$\begin{aligned} P(Y \geq 8) &= P(Y = 8) + P(Y = 9) + P(Y = 10) \\ &= \frac{26 + 3 + 1}{5000} \\ &= 0.006. \end{aligned}$$

(c)

$$\begin{aligned} P(2 <= Y < 8) &= P(Y < 8) - P(Y < 2) \\ &= [1 - P(Y \geq 8)] - P(Y = 1) \\ &= (1 - 0.006) - \frac{90}{5000} \\ &= 0.976 \end{aligned}$$

4. Let B be the event that the fruit fly is colored black. Then

$$P(B) = 0.35 \qquad \text{and} \qquad P(B^c) = 0.65.$$

Let Y : The number of 3 randomly selected fruit flies that are black. Then Y has probability distribution:

Y (Number of flies)	$P(Y = y)$
0	0.275
1	0.444
2	0.239
3	0.043
Total	1.000

(a)

$$\begin{aligned} P(Y \geq 1) &= 1 - P(Y < 1) \\ &= 1 - P(Y = 0) \\ &= 1 - 0.275 \\ &= 0.725 \end{aligned}$$

(b)

$$\begin{aligned} P(Y < 3) &= 1 - P(Y \geq 3) \\ &= 1 - P(Y = 3) \\ &= 1 - 0.043 \\ &= 0.957 \end{aligned}$$

(c)

$$
\begin{aligned}
E(Y) &= \sum_{y=0}^{3} y \cdot P(Y = y) \\
&= 0 \cdot 0.275 + 1 \cdot 0.444 + 2 \cdot 0.239 + 3 \cdot 0.043 \\
&= 1.051
\end{aligned}
$$

5. (a) Let A : Outcome of a fair coin flip (Heads = 1, Tails = 0), and let B : Outcome of a fair dice roll. By simulating 100 coin flips and dice rolls we obtain the following probability distribution for the outcome of our experiment (Y):

Y	Distribution	Data
1	0.16667	0.12
2	0.16667	0.14
3	0.16667	0.18
4	0.16667	0.19
5	0.16667	0.2
6	0.16667	0.17

First note that

$$
\begin{aligned}
P(B = 4) \cdot P(A = 0) & \\
&= 0.19 \cdot 0.5 \\
&= 0.095 \\
&\approx 0.0833 \\
&= P(B = 4 | A = 0) P(A = 0) \\
&= P(A = 0 \cap B = 4)
\end{aligned}
$$

Since our simulation shows that $P(A = 0 \cap B = 4)$ is close to $P(A = 0) \cdot P(B = 4)$ then our data provide evidence that A and B are theoretically independent from one another.

(b) We now let A : Outcome of a fair coin flip (Heads = 1, Tails = 0), and let B : Outcome of a dice roll where the probabilities of the dice roll change depending on the outcome of the coin flip. That is

$$P(B = b \mid A = 1) = \begin{cases} 0.167, & \text{when } b = 1 \\ 0.167, & \text{when } b = 2 \\ 0.167, & \text{when } b = 3 \\ 0.167, & \text{when } b = 4 \\ 0.167, & \text{when } b = 5 \\ 0.167, & \text{when } b = 6 \end{cases}$$

$$P(B = b \mid A = 0) = \begin{cases} 0.1 & \text{when } b = 1 \\ 0.1, & \text{when } b = 2 \\ 0.1, & \text{when } b = 3 \\ 0.5, & \text{when } b = 4 \\ 0.1, & \text{when } b = 5 \\ 0.1, & \text{when } b = 6 \end{cases}.$$

By simulating 100 coin flips and dice rolls we obtain the following probability distribution for our experiment (Y):

Y	Distribution	Data
1	0.13333	0.1
2	0.13333	0.2
3	0.13333	0.13
4	0.33333	0.32
5	0.13333	0.08
6	0.13333	0.17

Now, since

$$\begin{aligned} P(A = 0 \cap B = 4) &= P(B = 4|A = 0)P(A = 0) \\ &= 0.5 \cdot 0.5 \\ &= 0.25 \end{aligned}$$

but from out data, we have

$$\begin{aligned} P(A = 0) \cdot P(B = 4) &= 0.5 \cdot 0.32 \\ &= 0.16 \end{aligned}$$

then clearly A and B are not independent since

$$P(A = 0 \cap B = 4) = 0.25 \not\approx 0.16 = P(A = 0) \cdot P(B = 4).$$