Stat 100a, Introduction to Probability.

<u>Outline for the day:</u>

1. P(AA and full house) and P(Ad Kd and royal flush).
2. P(A♠ after first ace).
3. Daniel vs. Gus.
4. P(flop 3 of a kind).
5. P(eventually make 4 of a kind).
6. Bayes's rule.
7. Random variables.
8. cdf, pmf, and density.
9. Expected value.

Read through chapter 4.

For problem 2.4, consider a royal flush an example of a straight flush. That is, calculate P(straight flush or royal flush). ♠ ♣ ♥ ♦

1. P(you get dealt AA and flop a full house)

= P(you get dealt AA) * P(you flop a full house | AA)

= C(4,2) / C(52,2) * P(triplet or Axx | AA)

= 6/1326 * (12 * C(4,3) + 2*12*C(4,2))/C(50,3)

= .00433%.

P(you are dealt A♦ K♦ and flop a royal flush)? This relates to the unbreakable nuts hw question in a way.

= P(you get dealt A♦ K♦) * P(you flop a royal flush | you have A♦ K♦)

= P(you get dealt A♦ K♦) * P(flop contains Q♦ J♦ 10♦ | you have A♦ K♦)

= 1 / C(52,2) * 1/C(50,3)

= 1 / 25,989,600.

Deal til first ace appears. Let X = the *next* card after the ace.

P(X = A♠)? P(X = 2♣)?

2. Deal til first ace appears. Let X = the *next* card after the ace.

P(X = A♠)? P(X = 2♣)?

(a) How many permutations of the 52 cards are there?

52!

(b) How many of these perms. have A♠ right after the 1st ace?

(i) How many perms of the *other* 51 cards are there?

51!

(ii) For *each* of these, imagine putting the A♠ right after the 1st ace.

1:1 correspondence between permutations of the other 51 cards & permutations of 52 cards such that A♠ is right after 1st ace.

So, the answer to question (b) is 51!.

Answer to the overall question is 51! / 52! = 1/52.

Obviously, same goes for 2♣.

3. High Stakes Poker, Daniel vs. Gus.

Which is more likely, given no info about your cards:

* flopping 3 of a kind,

or

* eventually making 4 of a kind?

4. P(flop 3-of-a-kind)?

[including case where all 3 are on board, and *not including full houses*]

Key idea: forget order! Consider all combinations of your 2 cards and the flop.

Sets of 5 cards. Any such combo is equally likely! choose(52,5) different ones.

P(flop 3 of a kind) = # of different 3 of a kinds / choose(52,5)

How many different 3 of a kind combinations are possible?

13 * choose(4,3) different choices for the triple.

For each such choice, there are choose(12,2) choices left for the numbers on the other 2 cards, and for each of these numbers, there are 4 possibilities for its suit.

So, P(flop 3 of a kind) = 13 * choose(4,3) * choose(12,2) * 4 * 4 / choose(52,5)

~ 2.11%, or 1 in 47.3.

P(flop 3 of a kind or a full house) = 13 * choose(4,3) * choose(48,2) / choose(52,5)

~ 2.26%, or 1 in 44.3.

5. P(eventually make 4-of-a-kind)? [including case where all 4 are on board]

Again, just forget card order, and consider all collections of 7 cards.

Out of choose(52,7) different combinations, each equally likely, how many of them involve 4-of-a-kind?

13 choices for the 4-of-a-kind.

For each such choice, there are choose(48,3) possibilities for the other 3 cards.

So, P(4-of-a-kind) = 13 * choose(48,3) / choose(52,7) ~ 0.168%, or 1 in 595.

6. Bayes's rule.

Suppose that B_1, B_2 , B_n are disjoint events and that exactly one of them must occur.

Suppose you want $P(B_1 | A)$, but you only know $P(A | B_1)$, $P(A | B_2)$, etc.,

and you also know $P(B_1), P(B_2), \ldots, P(B_n)$.

Bayes' Rule: If $B_{1, \ldots,} B_n$ are disjoint events with $P(B_1 \text{ or } \ldots \text{ or } B_n) = 1$, then

$$P(B_i | A) = P(A | B_i) * P(B_i) \div [\sum P(A | B_j)P(B_j)].$$

Why? Recall: $P(X | Y) = P(X \& Y) \div P(Y)$. So $P(X \& Y) = P(X | Y) * P(Y)$.

$$P(B_1 | A) = P(A \& B_1) \div P(A)$$

$$= P(A \& B_1) \div [\ P(A \& B_1) + P(A \& B_2) + \ldots + P(A \& B_n)\ ]$$

$$= P(A | B_1) * P(B_1) \div [\ P(A | B_1)P(B_1) + P(A | B_2)P(B_2) + \ldots + P(A | B_n)P(B_n)\ ].$$

Bayes's rule, continued.

Bayes's rule: If $B_1, \ldots, B_n$ are disjoint events with $P(B_1 \text{ or } \ldots \text{ or } B_n) = 1$, then

$P(B_i \mid A) = P(A \mid B_i) * P(B_i) \div [\sum P(A \mid B_j)P(B_j)]$.

See example 3.4.1, p50. If a test is 95% accurate and 1% of the pop. has a condition, then given a random person from the population,

P(she has the condition | she tests positive)

= P(cond | +)

= P(+ | cond) P(cond) ÷ [P(+ | cond) P(cond) + P(+ | no cond) P(no cond)]

= 95% x 1% ÷ [95% x 1% + 5% x 99%]

~ 16.1%.

Tests for rare conditions must be extremely accurate.

Bayes' rule example.

Suppose P(your opponent has the nuts) = 1%, and P(opponent has a weak hand) = 10%.

Your opponent makes a huge bet. Suppose she'd only do that with the nuts or a weak hand, and that P(huge bet | nuts) = 100%, and P(huge bet | weak hand) = 30%.

What is P(nuts | huge bet)?

P(nuts | huge bet) =

$$\frac{P(\text{huge bet} \mid \text{nuts}) * P(\text{nuts})}{P(\text{huge bet} \mid \text{nuts})\, P(\text{nuts}) + P(\text{huge bet} \mid \text{horrible hand})\, P(\text{horrible hand})}$$

$$= \frac{100\% * 1\%}{100\% * 1\% + 30\% * 10\%}$$

= **25%.**

7. Random variables.

A *variable* is something that can take different numeric values.

A *random variable* (X) can take different numeric values with different probabilities.

X is *discrete* if all its possible values can be listed. If X can take any value in an interval like say [0,1], then X is *continuous*.

Ex. Two cards are dealt to you. Let X be 1 if you get a pair, and X is 0 otherwise.

P(X is 1) = 3/51 ~ 5.9%.   P(X is 0) ~ 94.1%.

Ex. A coin is flipped, and X=20 if heads, X=10 if tails.

The *distribution* of X means all the information about all the possible values X can take, along with their probabilities.

8. cdf, pmf, and density (pdf).

Any random variable has a *cumulative distribution function* (cdf):

$F(b) = P(X \leq b)$.

If X is discrete, then it has a *probability mass function* (pmf):

$f(b) = P(X = b)$.

Continuous random variables are often characterized by their *probability density functions* (pdf, or *density*):

a function f(x) such that $P(X \text{ is in } B) = \int_B f(x)\, dx$.

9. Expected Value.

For a discrete random variable X with pmf f(b), the *expected value* of X = $\sum$ b f(b).

The sum is over all possible values of b. (continuous random variables later…)

The expected value is also called the *mean* and denoted E(X) or μ.

Ex: 2 cards are dealt to you. X = 1 if pair, 0 otherwise.

P(X is 1) ~ 5.9%, P(X is 0) ~ 94.1%.

E(X) = (1 x 5.9%) + (0 x 94.1%) = 5.9%, or 0.059.

Ex. Coin, X=20 if heads, X=10 if tails.

E(X) = (20x50%) + (10x50%) = 15.

Ex. Lotto ticket. f($10million) = 1/choose(52,6) = 1/20million, f($0) = 1-1/20mil.

E(X) = ($10mil x 1/20million) = $0.50.

The expected value of X represents a *best guess* of X.

Compare with the *sample mean*, $\overline{X} = (X_1 + X_1 + \ldots + X_n) / n$.