Stat 100a: Introduction to Probability.

Outline for the day:

1. Hand in HW1. See hw2.
2. All in with 55.
3. Expected value and pot odds.
4. Pot odds example, Elezra and Violette.
5. P(flop 4 of a kind).
6. Variance and SD.
7. Markov and Chebyshev inequalities.
8. Luck and skill in poker.
9. Lederer and Minieri.

I will assign you to teams on Tuesday.

♠ ♣ ♥ ♦

1. Hand in HW1.

2. All in with 55?

a) You have $100 and 55 and are up against A9. You are 56% to win, so your expected value is $112.

b) You have $100 and 55 and are up against A9, KJ, and QJs. Seems pretty terrible, doesn't it? But you have a probability of 27.3% to quadruple, so your expected value is

0.273 x $400 = $109. About the same as #1!

[For these probabilities, see the online Texas Holdem odds calculator at http://www.cardplayer.com .]

2. Expected value and pot odds.

Suppose someone bets (or raises) you, going all-in. What should your chances of winning be in order for you to correctly call?

Let B = the amount bet to you, i.e. the additional amount you'd need to put in if you want to call. So, if you bet 100 & your opponent with 800 left went all-in, B = 700.

Let POT = the amount in the pot right now (including your opponent's bet).

Let p = your probability of winning the hand if you call. So prob. of losing = 1-p.

Let CHIPS = the number of chips you have right now.

If you call, then E[your chips at end] = (CHIPS - B)(1-p) + (CHIPS + POT)(p)

= CHIPS(1-p+p) - B(1-p) + POT(p) = CHIPS - B + Bp + POTp

If you fold, then E[your chips at end] = CHIPS.

You want your expected number of chips to be maximized, so it's worth calling if

-B + Bp + POTp > 0, i.e. if **p > B / (B+POT).**

The denominator is how large the pot will be after you call.

From previous slide, to call an all-in, need P(win) > B ÷ (B+pot).
Expressed as an *odds ratio*, this is sometimes referred to as *pot odds* or *express odds*.

If the bet is not all-in & another betting round is still to come, need
P(win) > wager ÷ (wager + winnings),
where winnings = pot + amount you'll win on later betting rounds,
wager = total amount you will wager including the current round & later rounds,
assuming no folding.

The terms *Implied-odds* / *Reverse-implied-odds* describe the cases where
winnings > pot or where *wager > B*, respectively. See p66.

You will not be tested on implied or reverse implied odds in this course.

Example: 2006 World Series of Poker (WSOP). ♠ ♣ ♥ ♦

Blinds: 200,000/400,000, + 50,000 ante.

Jamie Gold (4♠ 3♣): 60 million chips. Calls.

Paul Wasicka (8♠ 7♠): 18 million chips. Calls.

Michael Binger (A♦ 10♦): 11 million chips. Raises to $1,500,000.

Gold & Wasicka call. (pot = 4,650,000) Flop: 6♠ 10♣ 5♠.

- Wasicka checks, Binger bets $3,500,000. (pot = 8,150,000)
- Gold moves all-in for 16,450,000. (pot = 24,600,000)
- Wasicka folds. Q: Based on expected value, should he have called?

If Binger will fold, then Wasicka's chances to beat Gold must be at least

16,450,000 / (24,600,000 + 16,450,000) = 40.1%.

If Binger calls, it's a bit complicated, but basically Wasicka's chances must be at

least 16,450,000 / (24,600,000 + 16,450,000 + 5,950,000) = 35.0%.

4. Pot odds example, *Poker Superstars Invitational Tournament*, FSN, October 2005.

Ted Forrest: 1 million chips
Freddy Deeb: 825,000 Blinds: 15,000 / 30,000
Cindy Violette: 650,000
Eli Elezra: 575,000

* Elezra raises to 100,000
* Forrest folds.
* Deeb, the small blind, folds.
* Violette, the big blind with K♦ J♦, calls.

* The flop is: 2♦ 7♣ A♦

* Violette bets 100,000. (pot = 315,000).
* Elezra raises all-in to 475,000. (pot = 790,000).

So, it's 375,000 more to Violette. She folds.

Q: Based on expected value, should she have called?
Her chances must be at least 375,000 / (790,000 + 375,000) = 32%.

Violette has K◆ J◆.   The flop is: 2◆ 7♣ A◆.

Q: Based on expected value, should she have called?

Her chances must be at least 375,000 / (790,000 + 375,000) = 32%.

vs.	AQ: 38%.	AK: 37%	AA: 26%	77: 26%	A7: 31%
	A2: 34%	72: 34%	TT: 54%	T9: 87%	73: 50%

Harrington's principle: always assume at least a 10% chance that opponent is bluffing.

Bayesian approach: average all possibilities, weighting them by their likelihood.

Violette has K◆ J◆. The flop is: 2◆ 7♣ A◆.

Q: Based on expected value, should she have called?

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Harrington's principle: always assume at least a 10% chance that opponent is bluffing.

Bayesian approach: average all possibilities, weighting them by their likelihood.

Reality: Elezra had 7◆ 3♥. Her chances were 51%. Bad fold.

What was her prob. of winning (given just her cards and Elezra's, and the flop)?

Of choose(45,2) = 990 combinations for the turn & river, how many give her the win?

First, how many outs did she have? eight ◆s + 3 kings + 3 jacks = 14.

She wins with (out, out) or (out, nonout) or (non-◆ Q, non-◆ T)

choose(14,2) + 14 x 31 + 3 * 3 = 534

but not (k or j, 7 or non-◆ 3) and not (3◆ , 7 or non-◆ 3)

- 6 * 4 - 1 * 4 = 506.

So the answer is 506 / 990 = 51.1%.

5. P(flop 4 of a kind).

Suppose you're all in next hand, no matter what cards you get.

P(flop 4 of a kind) = 13*48 / choose(52,5) = 0.024% = 1 in **4165**.

P(flop 4 of a kind | pocket pair)?

No matter which pocket pair you have, there are choose(50,3) possible flops, each equally likely, and how many of them give you 4-of-a-kind?

48. (e.g. if you have 7♠ 7♥, then need to flop 7♦ 7♣ x, & there are 48 choices for x)

So P(flop 4-of-a-kind | pp) = 48/choose(50,3) = 0.245% = 1 in **408**.

6. Variance and SD.

Expected Value: $E(X) = \mu = \sum k\, P(X=k)$.

Variance: $V(X) = \sigma^2 = E[(X- \mu)^2]$. Turns out this $= E(X^2) - \mu^2$.

Standard deviation $= \sigma = \sqrt{V(X)}$. Indicates how far an observation would *typically* deviate from μ.

Examples:

Game 1. Say X = \$4 if red card, X = \$-5 if black.

E(X) = (\$4)(0.5) + (\$-5)(0.5) = -\$0.50.

$E(X^2)$ = (\$$4^2$)(0.5) + (\$-5^2)(0.5) = (\$16)(0.5) + (\$25)(0.5) = \$20.5.

So $\sigma^2 = E(X^2) - \mu^2$ = \$20.5 - \$-0.50^2 = \$20.25. **σ = \$4.50**.

Game 2. Say X = \$1 if red card, X = \$-2 if black.

E(X) = (\$1)(0.5) + (\$-2)(0.5) = -\$0.50.

$E(X^2)$ = (\$$1^2$)(0.5) + (\$-2^2)(0.5) = (\$1)(0.5) + (\$4)(0.5) = \$2.50.

So $\sigma^2 = E(X^2) - \mu^2$ = \$2.50 - \$-0.50^2 = \$2.25. **σ = \$1.50**.

7. Markov and Chebyshev inequalities. Ch 4.6.

The Markov inequality states

If X takes only non-negative values, and c is any number > 0, then

$P(X \geq c) \leq E(X)/c$.

Proof. The discrete case is given on p82.

Here is a proof for the case where X is continuous with pdf f(y).

$E(X) = \int y\, f(y)\, dy$

$= \int_0^c y f(y) dy + \int_c^\infty y f(y) dy$

$\geq \int_c^\infty y f(y) dy$

$\geq \int_c^\infty c f(y) dy$

$= c \int_c^\infty f(y) dy$

$= c\, P(X \geq c)$.

Thus, $P(X \geq c) \leq E(X) / c$.

The Chebyshev inequality states

For any random variable Y with expected value μ and variance σ^2, and any real number $a > 0$,

$P(| Y - \mu | \geq a) \leq \sigma^2 / a^2$.

Proof. The Chebyshev inequality follows directly from the Markov equality by letting $c = a^2$ and $X = (Y-\mu)^2$.

Examples of the use of the Markov and Chebyshev inequalities are on p83.

8. Luck and skill in poker. ♠ ♣ ♥ ♦

Let equity = your expected portion of the pot after the hand, assuming no future betting.
= your expected number of chips after the hand - chips you had before the hand
= pot * p, where p = your probability of winning if nobody folds.
I define luck as the equity gained during the dealing of the cards.
Skill = equity gained during the betting rounds.

Example.
You have Q♣ Q♦. I have 10♠ 9♠. Board is 10♦ 8♣ 7♣ 4♣. Pot is $5.
The river is 2♦, you bet $3, and I call.
On the river, how much equity did you gain by luck and how much by skill?

Equity gained by luck on river = your equity when 2♦ is exposed – your equity on turn
= 100% ($5) - 35/44 ($5) = $1.02.
Why 35/44? I can win with a 10, 9, 6, or J that is not a club. There are 1 + 2 + 3 + 3 = 9 of these cards, so the remaining 35 cards give you the win.

Equity gained by skill on river = your equity after all the betting is over - your equity when the 2♦ is dealt
= increase in pot on river * P(you win) - your cost
= $6 * 100% - $3 = $3.

Luck and skill in poker, continued. ♠ ♣ ♥ ♦

Example.

You have Q♣ Q♦. I have 10♠ 9♠. Board is 10♦ 8♣ 7♣ 4♣. Pot is $5.

The river is 2♦, you bet $3, and I call.

On the river, how much equity did you gain by luck and how much by skill?

Alternatively, let x = the number of chips you have after your $3 bet on the river.

Before this bet, you had x + $3 chips.

Equity gained by skill on river = your equity after all the betting is over - your equity when the 2♦ is dealt

= your expected number of chips after all the betting is over – your expected number of chips when the 2d is dealt

= (100%)(x + $11) – (100%)(x + $3 + $5)

= $3.

9. Lederer and Minieri.
2 min into https://www.youtube.com/watch?v=-MbLOWXPaNM .
3 min in https://www.youtube.com/watch?v=4pU68XVCuaU .

Let equity = your expected portion of the pot after the hand, assuming no future betting.
= your expected number of chips after the hand - chips you had before the hand
= pot * p, where p = your probability of winning if nobody folds.
I define luck as the equity gained during the dealing of the cards.
Skill = equity gained during the betting rounds.

Are there any problems with these definitions?

10. Bluffing. Ivey and Booth.

8. Facts about expected value.

For any random variable X and any constants a and b,

$E(aX + b) = aE(X) + b$.

Also, $E(X+Y) = E(X) + E(Y)$,

unless $E(X) = \infty$ and $E(Y) = -\infty$, in which case $E(X)+E(Y)$ is undefined.

Thus $\sigma^2 = E[(X-\mu)^2]$

$= E[(X^2 - 2\mu X + \mu^2)]$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$.