Stat 100a, Introduction to Probability.

Outline for the day:

1. Geometric random variables.
2. Negative binomial random variables.
3. Moment generating functions.
4. Poisson random variables.
5. Continuous random variables and their densities.
6. Uniform random variables.
7. Exponential random variables.
8. Harman/Negreanu and running it twice.

HW2 is due Nov7. The midterm is Tue Nov 7 in class. There is no lecture Thu Nov 9.

http://www.stat.ucla.edu/~frederic/100a/F17

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1. Geometric random variables, ch 5.3.

Suppose now X = # of trials until the *first* occurrence.

(Again, each trial is independent, and each time the probability of an occurrence is p.)

Then X = *Geometric (p)*.

e.g. the number of hands til you get your next pocket pair.

[*Including the hand where you get the pocket pair. If you get it right away, then X = 1.*]

Now X could be 1, 2, 3, …, up to ∞.

pmf: $P(X = k) = p^1 q^{k-1}$.

e.g. say k=5: $P(X = 5) = p^1 q^4$. Why? Must be 0 0 0 0 1. Prob. = q * q * q * q * p.

If X is Geometric (p), then $\mu = 1/p$, and $\sigma = (\sqrt{q}) \div p$.

e.g. Suppose X = the number of hands til your next pocket pair. P(X = 12)? E(X)? σ?

X = Geometric (5.88%).

$P(X = 12) = p^1 q^{11}$ = 0.0588 * 0.9412 ^ 11 = **3.02%.**

E(X) = 1/p = **17.0**. σ = sqrt(0.9412) / 0.0588 = **16.5**.

So, you'd typically *expect* it to take 17 hands til your next pair, +/- around 16.5 hands.

2. Negative binomial random variables, ch5.4.

Recall: if each trial is independent, and each time the probability of an occurrence is p, and X = # of trials until the *first* occurrence, then:

X is Geometric (p), $P(X = k) = p^1 q^{k-1}$, $\mu = 1/p$, $\sigma = (\sqrt{q}) \div p$.

Suppose now X = # of trials until the *rth* occurrence.

Then X = *negative binomial (r,p)*.

e.g. the number of hands you have to play til you've gotten r=3 pocket pairs.

Now X could be 3, 4, 5, …, up to ∞.

pmf: $P(X = k) = \text{choose}(k-1, r-1)\, p^r q^{k-r}$, for k = r, r+1, ….

e.g. say r=3 & k=7: $P(X = 7) = \text{choose}(6,2)\, p^3 q^4$.

Why? Out of the first 6 hands, there must be exactly r-1 = 2 pairs. Then pair on 7th.

P(exactly 2 pairs on first 6 hands) = $\text{choose}(6,2)\, p^2 q^4$. P(pair on 7th) = p.

If X is negative binomial (r,p), then $\mu = r/p$, and $\sigma = (\sqrt{rq}) \div p$.

e.g. Suppose X = the number of hands til your 12th pocket pair. P(X = 100)? E(X)? σ?

X = Neg. binomial (12, 5.88%).

$P(X = 100) = \text{choose}(99,11)\, p^{12} q^{88}$

= choose(99,11) * 0.0588 ^ 12 * 0.9412 ^ 88 = **0.104%.**

E(X) = r/p = 12/0.0588 ~ **204**. σ = sqrt(12*0.9412) / 0.0588 = **57.2**.

So, you'd typically *expect* it to take 204 hands til your 12th pair, +/- around 57.2 hands.

3. Moment generating functions, ch. 4.7

Suppose X is a random variable. E(X), $E(X^2)$, $E(X^3)$, etc. are the *moments* of X.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Take derivatives with respect to t of $ø_X(t)$ and evaluate at t=0 to get moments of X.

1st derivative $(d/dt)\, e^{tX} = X\, e^{tX}$, $\quad (d/dt)^2\, e^{tX} = X^2\, e^{tX}$, etc.

$(d/dt)^k E(e^{tX}) = E[(d/dt)^k e^{tX}] = E[X^k e^{tX}]$, (see p.84)

so $ø'_X(0) = E[X^1 e^{0X}] = E(X)$,

$ø''_X(0) = E[X^2 e^{0X}] = E(X^2)$, etc.

The moment gen. function $ø_X(t)$ uniquely characterizes the distribution of X.

So to show that X is, say, Poisson, you just need to show that it has the moment generating function of a Poisson random variable.

Also, if X_i are random variables with cdfs F_i, and $ø_{X_i}(t) \to ø(t)$, where $ø_X(t)$ is the moment generating function of X which has cdf F, then $X_i \to X$ in distribution, i.e. $F_i(y) \to F(y)$ for all y where F(y) is continuous, see p85.

Moment generating functions, continued.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Suppose X is Bernoulli (0.4). What is $ø_X(t)$?

$E(e^{tX}) = (0.6)\ (e^{t(0)}) + (0.4)\ (e^{t(1)}) = 0.6 + 0.4\ e^t$.

Suppose X is Bernoulli (0.4) and Y is Bernoulli (0.7) and X and Y are independent.

What is the distribution of XY?

$ø_{XY}(t) = E(e^{tXY}) = P(XY=0)\ (e^{t(0)}) + P(XY=1)(e^{t(1)})$

$= P(X=0 \text{ or } Y=0)\ (1) + P(X=1 \text{ and } Y=1)e^t$

$= [1 – P(X=1)P(Y=1)] + P(X=1)P(Y=1)e^t$

$= [1 – 0.4 \text{ x } 0.7] + 0.4\text{x}0.7e^t$

$= 0.72 + 0.28e^t$, which is the moment generating function of a Bernoulli (0.28) random variable. Therefore XY is Bernoulli (0.28).

What about $Z = \min\{X,Y\}$?

Z = XY in this case, since X and Y are 0 or 1, so the answer is the same.

4. Poisson random variables, ch 5.5.

Player 1 plays in a very slow game, 4 hands an hour, and she decides to do a big bluff whenever the second hand on her watch, at the start of the deal, is in some predetermined 10 second interval.

Now suppose Player 2 plays in a game where about 10 hands are dealt per hour, so he similarly looks at his watch at the beginning of each poker hand, but only does a big bluff if the second hand is in a 4 second interval.

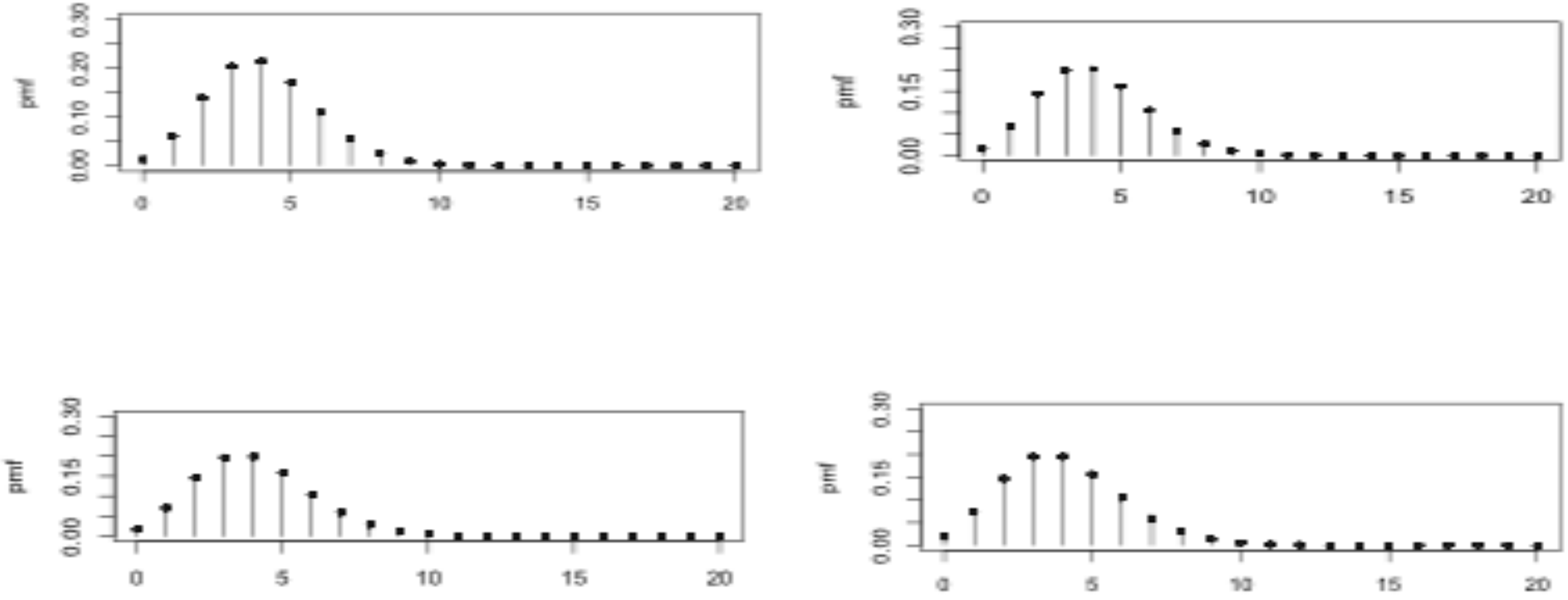
Player 3 plays in a faster game where about 20 hands are dealt per hour, and she bluffs only when the second hand on her watch at the start of the deal is in a 2 second interval. Each of the three players will thus average one bluff every hour and a half.

Let X_1, X_2, and X_3 denote the number of big bluffs attempted in a given 6 hour interval by Player 1, Player 2, and Player 3, respectively.

Each of these random variables is binomial with an expected value of 4, and a variance approaching 4.

They are converging toward some limiting distribution, and that limiting distribution is called the *Poisson* distribution.

They are converging toward some limiting distribution, and that limiting distribution is called the *Poisson* distribution. Unlike the binomial distribution which depends on two parameters, n and p, the Poisson distribution depends only on one parameter, λ, which is called the *rate*. In this example, $\lambda = 4$.



The pmf of the Poisson random variable is $f(k) = e^{-\lambda}\lambda^k/k!$, for $k=0,1,2,...$, and for $\lambda > 0$, with the convention that 0!=1, and where $e = 2.71828...$.
The Poisson random variable is the limit in distribution of the binomial distribution as $n \to \infty$ while np is held constant.

For a Poisson(λ) random variable X, $E(X) = \lambda$, and $Var(X) = \lambda$ also. $\lambda = rate$.

Example. Suppose in a certain casino jackpot hands are defined so that they tend to occur about once every 50,000 hands on average. If the casino deals approximately 10,000 hands per day, **a)** what are the expected value and standard deviation of the number of jackpot hands dealt in a 7 day period? **b)** How close are the answers using the binomial distribution and the Poisson approximation? Using the Poisson model, if X represents the number of jackpot hands dealt over this week, what are **c)** $P(X = 5)$ and **d)** $P(X = 5 \mid X > 1)$?

Answer. It is reasonable to assume that the outcomes on different hands are iid, and this applies to jackpot hands as well. In a 7 day period, approximately 70,000 hands are dealt, so X = the number of occurrences of jackpot hands is binomial$(n=70{,}000, p=1/50{,}000)$. Thus **a)** $E(X) = np = 1.4$, and $SD(X) = \sqrt{(npq)} = \sqrt{(70{,}000 \times 1/50{,}000 \times 49{,}999/50{,}000)} \sim 1.183204$. **b)** Using the Poisson approximation, $E(X) = \lambda = np = 1.4$, and $SD(X) = \sqrt{\lambda} \sim 1.183216$. The Poisson model is a very close approximation in this case. Using the Poisson model with rate $\lambda = 1.4$,

c) $P(X=5) = e^{-1.4}\, 1.4^5/5! \sim 1.105\%$.

d) $P(X = 5 \mid X > 1) = P(X = 5 \text{ and } X > 1) \div P(X > 1) = P(X = 5) \div P(X>1) = [e^{-1.4}\ 1.4^5/5!] \div [1 - e^{-1.4}\ 1.4^0/0! - e^{-1.4}\, 1.4^1/1!] \sim 2.71\%$.

5. Continuous random variables and their densities, ch6.1.

Density (or pdf = Probability Density Function) f(y):

$$\int_B f(y)\, dy = P(X \text{ in } B).$$

Expected value, $\mu = E(X) = \int y\, f(y)\, dy$. (= $\sum y\, P(y)$ for discrete X.)

Variance, $\sigma^2 = V(X) = E(X^2) - \mu^2$.

$SD(X) = \sqrt{V(X)}$.

For examples of pdfs, see p104, 106, and 107.

6. Uniform Random Variables and R, ch6.3.

Continuous random variables are often characterized by their *probability density functions* (pdf, or *density*): a function f(x) such that P{X is in B} = $\int_B f(x)\,dx$.

Uniform: f(x) = c, for x in (a, b).

= 0, for all other x.

[Note: c must = 1/(b-a), so that $\int_a^b f(x)\,dx$ = P{X is in (a,b)} = 1.]

Uniform (0,1). See p107-109.

f(y) = 1, for y in (0,1). $\mu = 0.5$. $\sigma \sim 0.29$.

P(X is between 0.4 and 0.6) = $\int_{.4}^{.6} f(y)\,dy = \int_{.4}^{.6} 1\,dy = 0.2$.

In R, runif(1,min=a,max=b) produces a pseudo-random uniform.

Uniform example.

For a continuous random variable X,
The pdf f(y) is a function where $\int_a^b f(y)dy = P\{X \text{ is in } (a,b)\}$,
$E(X) = \mu = \int_{-\infty}^{\infty} y\, f(y)dy$,
and $\sigma^2 = Var(X) = E(X^2) - \mu^2$. $sd(X) = \sigma$.

For example, suppose X and Y are independent uniform random variables on (0,1), and Z = *min*(X,Y). **a)** Find the pdf of Z. **b)** Find E(Z). **c)** Find SD(Z).

a. For c in (0,1), $P(Z > c) = P(X > c \;\&\; Y > c) = P(X > c)\, P(Y > c) = (1-c)^2 = 1 – 2c + c^2$.
So, $P(Z \le c) = 1 – (1 – 2c + c^2) = 2c - c^2$.
Thus, $\int_0^c f(c)dc = 2c - c^2$. So f(c) = the derivative of $2c – c^2 = 2 – 2c$, for c in (0,1).
Obviously, f(c) = 0 for all other c.
b. $E(Z) = \int_{-\infty}^{\infty} y\, f(y)dy = \int_0^1 c\,(2-2c)\, dc = \int_0^1 2c – 2c^2\, dc = c^2 – 2c^3/3]_{c=0}^{1}$
$= 1 – 2/3 – (0 – 0) = 1/3$.
c. $E(Z^2) = \int_{-\infty}^{\infty} y^2 f(y)dy = \int_0^1 c^2 (2-2c)\, dc = \int_0^1 2c^2 – 2c^3\, dc = 2c^3/3 – 2c^4/4]_{c=0}^{1}$
$= 2/3 – 1/2 – (0 – 0) = 1/6$.
So, $\sigma^2 = Var(Z) = E(Z^2) – [E(Z)]^2 = 1/6 – (1/3)^2 = 1/18$.
$SD(Z) = \sigma = \sqrt{(1/18)} \sim 0.2357$.

7. Exponential distribution, ch 6.4.

Useful for modeling waiting times til something happens (like the geometric).

pdf of an exponential random variable is $f(y) = \lambda\ exp(-\lambda\ y)$, for $y \geq 0$, and $f(y) = 0$ otherwise.
The cdf is $F(y) = 1 - exp(-\lambda\ y)$, for $y \geq 0$.
If X is exponential with parameter λ, then $E(X) = SD(X) = 1/\lambda$

If the total numbers of events in any disjoint time spans are independent, then these totals are Poisson random variables. If in addition the events are occurring at a constant rate λ, then the times between events, or *interevent times*, are exponential random variables with mean $1/\lambda$.

Example. Suppose you play 20 hands an hour, with each hand lasting exactly 3 minutes, and let X be the time in hours until the end of the first hand in which you are dealt pocket aces. Use the exponential distribution to approximate $P(X \leq 2)$ and compare with the exact solution using the geometric distribution.

Answer. Each hand takes 1/20 hours, and the probability of being dealt pocket aces on a particular hand is 1/221, so the rate λ = 1 in 221 hands = 1/(221/20) hours ~ 0.0905 per hour.
Using the exponential model, $P(X \le 2 \text{ hours}) = 1 - \exp(-2\lambda) \sim 16.556\%$.
This is an approximation, however, since by assumption *X* is not continuous but must be an integer multiple of 3 minutes.
Let *Y* = the number of hands you play until you are dealt pocket aces. Using the geometric distribution, $P(X \le 2 \text{ hours}) = P(Y \le 40 \text{ hands})$
$= 1 - (220/221)^{40} \sim 16.590\%$.

The survivor function for an exponential random variable is particularly simple: $P(X > c) = \int_c^\infty f(y)dy = \int_c^\infty \lambda \exp(-\lambda y)dy = -\exp(-\lambda y)]_c^\infty = \exp(-\lambda c)$.

Like geometric random variables, exponential random variables have the *memorylessness* property: if *X* is exponential, then for any non-negative values *a* and *b*, $P(X > a+b \mid X > a) = P(X > b)$. *(See p115).*
Thus, with an exponential (or geometric) random variable, if after a certain time you still have not observed the event you are waiting for, then the distribution of the *future,* additional waiting time until you observe the event is the same as the distribution of the *unconditional* time to observe the event to begin with.

Harman / Negreanu, and running it twice.

Harman has 10♠ 7♠ . Negreanu has K♥ Q♥ . The flop is 10◆ 7♣ K◆ .

Harman's all-in. $156,100 pot. P(Negreanu wins) = 28.69%. P(Harman wins) = 71.31%.

Let X = amount Harman has after the hand.

If they run it once, E(X) = $0 x 29% + $156,100 x 71.31% = **$111,314.90.**

<u>If they run it twice, what is E(X)?</u>

There's some probability p_1 that Harman wins both times ==> X = $156,100.

There's some probability p_2 that they each win one ==> X = $78,050.

There's some probability p_3 that Negreanu wins both ==> X = $0.

E(X) = $156,100 x p_1 + $78,050 x p_2 + $0 x p_3.

If the different runs were *independent*, then p_1 = P(Harman wins 1st run & 2nd run) would = P(Harman wins 1st run) x P(Harman wins 2nd run) = 71.31% x 71.31% ~ 50.85%.

But, they're not quite independent! Very hard to compute p_1 and p_2.

However, you don't need p_1 and p_2 !

X = the amount Harman gets from the 1st run + amount she gets from 2nd run, so

E(X) = E(amount Harman gets from 1st run) + E(amount she gets from 2nd run)

= $78,050 x P(Harman wins 1st run) + $0 x P(Harman loses first run)

+ $78,050 x P(Harman wins 2nd run) + $0 x P(Harman loses 2nd run)

= $78,050 x 71.31% + $0 x 28.69% + $78,050 x 71.31% + $0 x 28.69% = **$111,314.90.**

HAND RECAP Harman 10♠ 7♠ Negreanu K♥ Q♥ The flop is 10♦ 7♣ K♦ .

Harman's all-in. $156,100 pot.P(Negreanu wins) = 28.69%. P(Harman wins) = 71.31%.

The standard deviation (SD) changes a lot! **Say they run it once.** (see p127.)

$V(X) = E(X^2) - \mu^2$.

μ = $111,314.9, so $\mu^2 \sim$ $12.3 billion.

$E(X^2)$ = ($156,100^2)(71.31%) + ($0^2$)(28.69%) = $17.3 billion.

V(X) = $17.3 billion - $12.3 bill. = $5.09 billion. SD σ = sqrt($5.09 billion)~$71,400.

So if they run it once, Harman expects to get back about $111,314.9 **+/- $71,400**.

If they run it twice? Hard to compute, but approximately, if each run were independent, then $V(X_1+X_2) = V(X_1) + V(X_2)$,

so if X_1 = amount she gets back on 1st run, and X_2 = amount she gets from 2nd run,

then $V(X_1+X_2) \sim V(X_1) + V(X_2) \sim$ $1.25 billion + $1.25 billion = $2.5 billion,

The standard deviation σ = sqrt($2.5 billion) ~ $50,000.

So if they run it twice, Harman expects to get back about $111,314.9 **+/- $50,000.**