Stat 100a: Introduction to Probability.

Outline for the day

0. Quick facts about normals.
1. Chip proportions and induction.
2. Doubling up.
3. Examples.

0. If X and Y are independent and both are normal, then X+Y is normal, and so are –X and –Y.

The computer project is due on Sat Dec2 8:00pm.

HW3 is due Tue Dec 5.

Thu Dec 7 is the final exam, here in class, 11am to 12:15pm.

Again any notes and books are fine, and bring a pencil and a calculator.

Also bring your student ID to the exam.

1. Chip proportions and induction, Theorem 7.6.6.

P(win a tournament) is proportional to your number of chips.

Simplified scenario. Suppose you either go up or down 1 each hand, with prob. 1/2.

Suppose there are n chips, and you have k of them.

Let p_k = P(win tournament given k chips) = P(random walk goes k -> n before hitting 0).

Now, clearly $p_0 = 0$. Consider p_1. From 1, you will either go to 0 or 2.

So, $p_1 = 1/2\ p_0 + 1/2\ p_2 = 1/2\ p_2$. That is, $p_2 = 2\ p_1$.

We have shown that $p_j = j\ p_1$, for j = 0, 1, and 2.

(induction:) **Suppose that, for j = 0, 1, 2, …, m, $p_j = j\ p_1$.**

We will show that $p_{m+1} = (m+1)\ p_1$.

Therefore, $p_j = j\ p_1$ for all j.

That is, P(win the tournament) is prop. to your number of chips.

$p_m = 1/2\ p_{m-1} + 1/2\ p_{m+1}$. If $p_j = j\ p_1$ for $j \le m$, then we have

$mp_1 = 1/2\ (m-1)p_1 + 1/2\ p_{m+1}$,

so $p_{m+1} = 2mp_1 - (m-1)\ p_1 = (m+1)p_1$.

2. Doubling up. Again, P(winning) = your proportion of chips.
Theorem 7.6.7, p152, describes another simplified scenario.
Suppose you either double each hand you play, or go to zero, each with probability 1/2.

Again, P(win a tournament) is prop. to your number of chips.

Again, $p_0 = 0$, and $p_1 = 1/2\ p_2 = 1/2\ p_2$, so again, $p_2 = 2\ p_1$.

We have shown that, for j = 0, 1, and 2, $p_j = j\ p_1$.

***(induction:)* Suppose that, for $j \leq m$, $p_j = j\ p_1$.**

We will show that $p_{2m} = (2m)\ p_1$.

Therefore, $p_j = j\ p_1$ for all $j = 2^k$. That is, P(win the tournament) is prop. to # of chips.

This time, $p_m = 1/2\ p_0 + 1/2\ p_{2m}$. If $p_j = j\ p_1$ for $j \leq m$, then we have

$mp_1 = 0 + 1/2\ p_{2m}$, so $p_{2m} = 2mp_1$. Done.

In Theorem 7.6.8, p152, you have k of the n chips in play. Each hand, you gain 1 with prob. p, or lose 1 with prob. q=1-p.

Suppose $0<p<1$ and $p \neq 0.5$. Let r = q/p. Then P(you win the tournament) = $\mathbf{(1-r^k)/(1-r^n)}$.

The proof is again by induction, and is similar to the proof we did of Theorem 7.6.6.

3. Examples.

(Chen and Ankenman, 2006). Suppose that a \$100 winner-take-all tournament has $1024 = 2^{10}$ players. So, you need to double up 10 times to win. Winner gets \$102,400.

Suppose you have probability $p = 0.54$ to double up, instead of 0.5.

What is your expected profit in the tournament? (Assume only doubling up.)

Answer. P(winning) = 0.54^{10}, so exp. return = 0.54^{10} (\$102,400) = \$215.89. So exp. <u>profit</u> = \$115.89.

What if each player starts with 10 chips, and you gain a chip with

p = 54% and lose a chip with p = 46%? What is your expected profit?

Answer. $r = q/p = .46/.54 = .852$. P(you win) = $(1\text{-}r^{10})/(1\text{-}r^{10240})$ = 79.9%.

So exp. profit = .799(\$102400) - \$100 ~ \$81700.

Random Walk example.

Suppose you start with 1 chip at time 0 and that your tournament is like a simple random walk, but if you hit 0 you are done. P(you have not hit zero by time 47)?

We know that starting at 0, $P(Y_1 \neq 0, Y_2 \neq 0, \ldots, Y_{2n} \neq 0) = P(Y_{2n} = 0)$.

So, $P(Y_1 > 0, Y_2 > 0, \ldots, Y_{48} > 0) = ½\ P(Y_{48} = 0) = ½$ Choose(48,24)$(½)^{48}$

$= P(Y_1 = 1, Y_2 > 0, \ldots, Y_{48} > 0)$

= P(start at 0 and win your first hand, and then stay above 0 for at least 47 more hands)

= P(start at 0 and win your first hand) x P(from (1,1), stay above 0 for ≥ 47 more hands)

= 1/2 P(starting with 1 chip, stay above 0 for at least 47 more hands).

So, P(starting with 1 chip, stay above 0 for at least 47 hands) = Choose(48,24)$(½)^{48}$

= 11.46%.

Bayes' rule example.

Your opponent raises all-in before the flop. Suppose you think she would do that *80%* of the time with AA, KK, or QQ, and she would do that *30%* of the time with AK or AQ, and 1% of the time with anything else.

Given only this, and not even your cards, what's P(she has AK)?

Given nothing, P(AK) = 16/C(52,2) = 16/1326. P(AA) = C(4,2)/C(52,2) = 6/1326.

Using Bayes' rule,

$$P(AK \mid \text{all-in}) = \frac{P(\text{all-in} \mid AK) * P(AK)}{P(\text{all-in}|AK)P(AK) + P(\text{all-in}|AA)P(AA) + P(\text{all-in}|KK)P(KK) + \dots},$$

$$= \frac{30\% \times 16/1326}{[30\%\text{x}16/1326] + [80\%\text{x}6/1326] + [80\%\text{x}6/1326] + [80\%\text{x}6/1326] + [30\%\text{x}16/1326] + [1\%\ (1326-16-6-6-6-16)/1326)]}.$$

(AK) (AA) (KK) (QQ) (AQ) (anything else)

= **13.06%.** Compare with 16/1326 ~ 1.21%.

Conditional probability examples.

Approximate P(SOMEONE has AA, given you have KK)? Out of your 8 opponents?

Note that given that you have KK,

P(player 2 has AA & player 3 has AA)

= P(player 2 has AA) x P(player 3 has AA | player 2 has AA)

= choose(4,2) / choose(50,2) x 1/choose(48,2)

= 0.0000043, or 1 in 230,000.

So, very little overlap! Given you have KK,

P(someone has AA) = P(player2 has AA or player3 has AA or … or pl.9 has AA)

~ P(player2 has AA) + P(player3 has AA) + … + P(player9 has AA)

= 8 x choose(4,2) / choose(50,2) = 3.9%, or 1 in **26.**

What is **exactly** P(someone has an Ace | you have KK)? (8 opponents)

or more than one ace

Given that you have KK, P(someone has an Ace) = 100% - P(nobody has an Ace).

And P(nobody has an Ace) = choose(46,16)/choose(50,16)

= **20.1%.**

So P(someone has an Ace) = **79.9%.**

Some other example problems.

a. Find the probability you are dealt a suited king.

4 * 12 / C(52,2) = 3.62%.

b. The typical number of hands until this occurs is ... +/- ....

1/.0362 ~ 27.6.
(√96.38%) / 3.62% ~ 27.1.
So the answer is 27.6 +/- 27.1.

More on luck and skill.

Gain due to skill on a betting round = your expected *profit* during the betting round
= your exp. chips after betting round – your exp. chips before betting round
= (equity after round + leftover chips) –
(equity before round + leftover chips + chips you put in during round)
= equity after round – equity before round – cost during round.

For example, suppose you have A♣ A♠, I have 3♥3♦, the board is
A♥ Q♣ 10♦ and there is $10 in the pot. The turn is 3♣
You go all in for $5 and I call. How much equity due to skill did you gain on the turn?
Your prob. of winning is 43/44.
Your skill gain on turn = your equity after turn bets - equity before turn bets – cost
= ($20)(43/44) - ($10)(43/44) - $5
= $4.77.

Suppose instead you bet $5 on the turn and I folded. How much equity due to skill did you gain on the turn?
$15(100%) - ($10)(43/44) - $5 = $0.23.

♠ ♣ ♥ ♦

P(You have AK | you have exactly one ace)?

= P(You have AK and exactly one ace) / P(exactly one ace)

= P(AK) / P(exactly one ace)

= (16/C(52,2)) ÷ (4x48/C(52,2))

= 4/48 = 8.33%.

P(You have AK | you have at least one ace)?

= P(You have AK and at least one ace) / P(at least one ace)

= P(AK) / P(at least one ace)

= (16/C(52,2)) ÷ (((4x48 + C(4,2))/C(52,2)) ~ 8.08%.

P(You have AK | your FIRST card is an ace)?

= 4/51 = 7.84%.

Suppose there are 1000 players in a casino. Each of them is playing holdem. What is the expected number of people who have pocket aces?

Let X_1 = 1 if player 1 has pocket aces, and 0 otherwise.

X_2 = 1 if player 2 has pocket aces, and 0 otherwise.

X_3 = 1 if player 3 has pocket aces, and 0 otherwise, etc.

X_1 and X_2 are not independent. Nevertheless, if Y = the number of people with AA,

then $Y = X_1 + X_2 + ... + X_{1000}$, and

$E(Y) = E(X_1) + E(X_2) + ... + E(X_{1000})$

$= C(4,2)/C(52,2) \times 1000$

~ 4.52.

Let X = the number of aces you have and Y = the number of kings you have. What is cov(X,Y)?

cov(X,Y) = E(XY) – E(X)E(Y).

X = $X_1 + X_2$, where X_1 = 1 if your first card is an ace and X_2 = 1 if your 2nd card is an ace,

so E(X) = E(X_1) + E(X_2) = 1/13 + 1/13 = 2/13. E(Y) = 2/13.

E(XY) = 1 if you have AK, and 0 otherwise, so E(XY) = 1 x P(AK) = 4x4/C(52,2) = .0121.

So, cov(X,Y) = .0121 – 2/13 x 2/13

= -.0116.

Another CLT Example

Suppose X1, X2, ..., X100 are 100 iid draws from a population with mean μ=70 and sd σ=10. What is the approximate distribution of the sample mean, $\bar{x}$?

By the CLT, the sample mean is approximately normal with mean μ and sd $\sigma/\sqrt{n}$, i.e. $\sim N(70, 1^2)$.

Now suppose Y1, Y2, ..., Y100 are iid draws, independent of X1, X2, X100, with mean μ=80 and sd σ=25. What is the approximate distribution of $\bar{x} - \bar{y} = Z$?

Now the sample mean of the first sample is approximately $N(70, 1^2)$ and similarly the negative sample mean of the 2nd sample is approximately $N(-80, 2.5^2)$, and the two are independent, so their sum Z is approximately normal.

Its mean is 70-80 = -10,

and $\text{var}(Z) = 1^2+2.5^2 = 7.25$, so $Z \sim N(-10, 2.69^2)$, because $2.69^2 = 7.25$.

Remember, if X and Y are ind., then var(X+Y) = var(X) + var(Y).