Stat 100a: Introduction to Probability.

<u>Outline for the day</u>

1. Hand in HW3.
2. Review list.
3. Examples.
4. Tournaments.

Thu Dec 7 is the final exam, here in class, 11am to 12:15pm.
Again any notes and books are fine, and bring a pencil and a calculator.
Also bring your student ID to the exam.

1. HAND IN HW3. 2. Review list.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. P(AB) = P(A) P(B|A) [= P(A)P(B) if ind.]
7) Odds ratios.
8) Random variables (RVs).
9) Discrete RVs, and probability mass function (pmf).
10) Expected value.
11) Pot odds calculations.
12) Luck and skill.
13) Variance and SD.
14) Bernoulli RV. [0-1. $\mu = p$, $\sigma = \sqrt{(pq)}$.]
15) Binomial RV. [# of successes, out of n tries. $\mu = np$, $\sigma = \sqrt{(npq)}$.]
16) Geometric RV. [# of tries til 1st success. $\mu = 1/p$, $\sigma = (\sqrt{q}) / p$.]
17) Negative binomial RV. [# of tries til rth success. $\mu = r/p$, $\sigma = (\sqrt{rq}) / p$.]
18) Poisson RV [# of successes in some time interval. [$\mu = \lambda$, $\sigma = \sqrt{\lambda}$.]
19) E(X+Y), V(X+Y) (ch. 7.1).
20) Bayes's rule (ch. 3.4).
20) Continuous RVs, Uniform, Normal, Exponential and Pareto.
21) Probability density function (pdf). Recall $F'(c) = f(c)$, where $F(c)$ = cdf.
22) Moment generating functions
23) Markov and Chebyshev inequalities
24) Law of Large Numbers (LLN) and Fundamental Theorem of Poker.
25) Central Limit Theorem (CLT)
26) Conditional expectation.
27) Confidence intervals for the sample mean and sample size calculations.
28) Random Walks, Reflection Principle, Ballot Theorem, avoiding zero
29) Chip proportions, doubling up, and induction.
30) Bivariate normal distribution and the conditional distribution of Y given X.
31) Covariance and correlation.

Basically, we've done all of ch. 1-7 except 6.7.

Bayes' rule example.

Your opponent raises all-in before the flop. Suppose you think she would do that *80%* of the time with AA, KK, or QQ, and she would do that *30%* of the time with AK or AQ, and 1% of the time with anything else.

Given only this, and not even your cards, what's P(she has AK)?

Given nothing, P(AK) = 16/C(52,2) = 16/1326. P(AA) = C(4,2)/C(52,2) = 6/1326.

Using Bayes' rule,

$$P(AK \mid \text{all-in}) = \frac{P(\text{all-in} \mid AK) * P(AK)}{P(\text{all-in}|AK)P(AK) + P(\text{all-in}|AA)P(AA) + P(\text{all-in}|KK)P(KK) + \dots},$$

$$= \frac{30\% \times 16/1326}{[30\% x16/1326] + [80\% x6/1326] + [80\% x6/1326] + [80\% x6/1326] + [30\% x16/1326] + [1\% (1326-16-6-6-6-16)/1326)]}.$$

(AK) (AA) (KK) (QQ) (AQ) (anything else)

= **13.06%.** Compare with 16/1326 ~ 1.21%.

Conditional probability examples.

Approximate P(SOMEONE has AA, given you have KK)? Out of your 8 opponents?

Note that given that you have KK,

P(player 2 has AA & player 3 has AA)

= P(player 2 has AA) x P(player 3 has AA | player 2 has AA)

= choose(4,2) / choose(50,2) x 1/choose(48,2)

= 0.0000043, or 1 in 230,000.

So, very little overlap! Given you have KK,

P(someone has AA) = P(player2 has AA or player3 has AA or … or pl.9 has AA)

~ P(player2 has AA) + P(player3 has AA) + … + P(player9 has AA)

= 8 x choose(4,2) / choose(50,2) = 3.9%, or 1 in **26.**

What is **exactly** P(someone has an Ace | you have KK)? (8 opponents)

or more than one ace

Given that you have KK, P(someone has an Ace) = 100% - P(nobody has an Ace).

And P(nobody has an Ace) = choose(46,16)/choose(50,16)

= **20.1%.**

So P(someone has an Ace) = **79.9%.**

Some other example problems.

a. Find the probability you are dealt a suited king.

4 * 12 / C(52,2) = 3.62%.

b. The typical number of hands until this occurs is ... +/- ....

1/.0362 ~ 27.6.
($\sqrt{96.38\%}$) / 3.62% ~ 27.1.
So the answer is 27.6 +/- 27.1.

More on luck and skill.

Gain due to skill on a betting round = your expected *profit* during the betting round
= your exp. chips after betting round – your exp. chips before betting round
= (equity after round + leftover chips) –
(equity before round + leftover chips + chips you put in during round)
= equity after round – equity before round – cost during round.

For example, suppose you have A♣ A♠, I have 3♥3◆, the board is
A♥ Q♣ 10◆ and there is $10 in the pot. The turn is 3♣.
You go all in for $5 and I call. How much equity due to skill did you gain on the turn?
Your prob. of winning is 43/44.
Your skill gain on turn = your equity after turn bets - equity before turn bets – cost
= ($20)(43/44) - ($10)(43/44) - $5
= $4.77.

Suppose instead you bet $5 on the turn and I folded. How much equity due to skill did you gain on the turn?
$15(100%) - ($10)(43/44) - $5 = $0.23.

♠ ♣ ♥ ♦

P(You have AK | you have exactly one ace)?

= P(You have AK and exactly one ace) / P(exactly one ace)

= P(AK) / P(exactly one ace)

= (16/C(52,2)) ÷ (4x48/C(52,2))

= 4/48 = 8.33%.

P(You have AK | you have at least one ace)?

= P(You have AK and at least one ace) / P(at least one ace)

= P(AK) / P(at least one ace)

= (16/C(52,2)) ÷ (((4x48 + C(4,2))/C(52,2)) ~ 8.08%.

P(You have AK | your FIRST card is an ace)?

= 4/51 = 7.84%.

Suspose there are 1000 players in a casino. Each of them is playing holdem. What is the expected number of people who have pocket aces?

Let X_1 = 1 if player 1 has pocket aces, and 0 otherwise.

X_2 = 1 if player 2 has pocket aces, and 0 otherwise.

X_3 = 1 if player 3 has pocket aces, and 0 otherwise, etc.

X_1 and X_2 are not independent. Nevertheless, if Y = the number of people with AA,

then $Y = X_1 + X_2 + ... + X_{1000}$, and

$E(Y) = E(X_1) + E(X_2) + ... + E(X_{1000})$

$= C(4,2)/C(52,2) \times 1000$

~ 4.52.

Let X = the number of queens you have and Y = the number of face cards you have. What is cov(X,Y)?

cov(X,Y) = E(XY) – E(X)E(Y).

$X = X_1 + X_2$, where $X_1 = 1$ if your first card is a queen and $X_2 = 1$ if your 2[nd] card is a queen,

so $E(X) = E(X_1) + E(X_2) = 1/13 + 1/13 = 2/13$. Similarly, E(Y) = 3/13 + 3/13 = 6/13.

E(XY)? XY = 4 if you have QQ, or 2 if you have KQ or QJ, or 1 if you have Qx, where x is not a face card,

or 0 otherwise. So E(XY) = 4 x C(4,2)/C(52,2) + 2 x (16+16)/C(52,2) + 1 x (4x40)/C(52,2) + 0 = 0.187.

So, cov(X,Y) = 0.187 – 2/13 x 6/13 =

= 0.116.

Another CLT Example

Suppose X1, X2, ..., X100 are 100 iid draws from a population with mean μ=70 and sd σ=10. What is the approximate distribution of the sample mean, $\bar{x}$?

By the CLT, the sample mean is approximately normal with mean μ and sd $\sigma/\sqrt{n}$, i.e. $\sim N(70, 1^2)$.

Now suppose Y1, Y2, ..., Y100 are iid draws, independent of X1, X2, X100, with mean μ=80 and sd σ=25. What is the approximate distribution of $\bar{x} - \bar{y} = Z$?

Now the sample mean of the first sample is approximately $N(70, 1^2)$ and similarly the negative sample mean of the 2nd sample is approximately $N(-80, 2.5^2)$, and the two are independent, so their sum Z is approximately normal.

Its mean is 70-80 = -10,

and $\mathrm{var}(Z) = 1^2+2.5^2 = 7.25$, so $Z \sim N(-10, 2.69^2)$, because $2.69^2 = 7.25$.

Remember, if X and Y are ind., then var(X+Y) = var(X) + var(Y).

Tournaments.