

# Stat 100a, Introduction to Probability.

## Outline for the day:

1. Computation of luck and skill.
2. Facts about expected value, exp number trick, cond. prob.
3. Bernoulli random variables.
4. Binomial random variables.
5. Geometric random variables.
6. Negative binomial random variables.
7. Moment generating functions.

There is no lecture Tue Nov5 because of the just announced faculty meeting. There is also no lecture or OH Thu Oct24.

Exam1 is Tue Oct29.

Read through chapter 5.

Homework 2 is on the course website. It is now due Thu Nov7.

Your emails are in the bottom of teams.txt, but I will take the emails off tonight.



<http://www.stat.ucla.edu/~frederic/100a/F19>

Mike Cloud raised to 15,000 with  $A\clubsuit A\spadesuit$ , Hellmuth called with  $A\heartsuit K\spadesuit$ , Daniel Negreanu called from the big blind with  $6\diamondsuit 4\heartsuit$ , and the flop came  $K\clubsuit 8\heartsuit K\heartsuit$ . Before the flop, the pot was 57,000 chips.

After the flop, all three players checked, the turn was the  $J\heartsuit$ , Negreanu checked, Cloud bet 15,000, Hellmuth called, and Negreanu folded.

The river was the  $7\spadesuit$ , Cloud checked, Hellmuth bet 37,000, and Cloud called. How much expected profit did Hellmuth gain due to luck and how much due to skill on the river?

Answer—When the turn was dealt, Hellmuth's probability of winning in a showdown was  $41/42 \sim 97.62\%$ . After the betting on the turn was over, the pot was 87,000 chips. When the  $7\spadesuit$  was revealed on the river, Hellmuth's equity increased from  $97.62\% \times 87,000 = 84,929.4$  to  $100\% \times 87,000$ , for an increase of 2070.6 chips due to luck.

Hellmuth's expected profit gained due to skill on the river is simply 37,000 chips: the pot size increased by 74,000 while Hellmuth had a 100% chance of winning, but the cost to Hellmuth was 37,000, so his profit was 37,000.

## 2. Facts about expected value.

For any random variable  $X$  and any constants  $a$  and  $b$ ,

$$E(aX + b) = aE(X) + b.$$

Also,  $E(X+Y) = E(X) + E(Y)$ ,

unless  $E(X) = \infty$  and  $E(Y) = -\infty$ , in which case  $E(X)+E(Y)$  is undefined.

Thus  $\sigma^2 = E[(X-\mu)^2]$

$$= E[(X^2 - 2\mu X + \mu^2)]$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2.$$

Expected number trick.

The board consists of 5 cards. Find the expected number of clubs on the board.

Let  $X_1 = 1$  if the 1<sup>st</sup> card is a club, and 0 otherwise.

Let  $X_2 = 1$  if the 2<sup>nd</sup> card is a club, and 0 otherwise.

etc.

$$X = X_1 + X_2 + X_3 + X_4 + X_5.$$

$$\text{So } E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5)$$

$$= \left[ \frac{1}{4} (1) + \frac{3}{4} (0) \right] \times 5 = 1.25.$$

Even though  $X_1, X_2, X_3, X_4$ , and  $X_5$  are not independent, nevertheless

$$E(X_1 + X_2 + X_3 + X_4 + X_5) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5).$$

Conditional probability.

When A and B are different outcomes on different collections of cards or different hands, then  $P(B|A)$  can often be found directly.

But when A and B are outcomes on the same event, or same card, then sometimes it is helpful to use the definition  $P(B|A) = P(AB)/P(A)$ .

For example, let A = the event your hole cards are black, and let B = the event your hole cards are clubs.

$$P(B|A) = P(AB)/P(A) = C(13,2)/C(52,2) / [C(26,2)/C(52,2)].$$

However, if A is the event your hole cards are black and B is the event the flop cards are all black, then  $P(B|A) = C(24,3)/C(50,3)$  directly.

### 3. Bernoulli Random Variables, ch. 5.1.

If  $X = 1$  with probability  $p$ , and  $X = 0$  otherwise, then  $X = \text{Bernoulli}(p)$ .

Probability mass function (pmf):

$$P(X = 1) = p$$

$$P(X = 0) = q, \quad \text{where } p+q = 100\%.$$

**If  $X$  is Bernoulli ( $p$ ), then  $\mu = E(X) = p$ , and  $\sigma = \sqrt{pq}$ .**

For example, suppose  $X = 1$  if you have a pocket pair next hand;  $X = 0$  if not.

$$p = 5.88\%. \quad \text{So, } q = 94.12\%.$$

[Two ways to figure out  $p$ :

(a) Out of  $\text{choose}(52,2)$  combinations for your two cards,  $13 * \text{choose}(4,2)$  are pairs.

$$13 * \text{choose}(4,2) / \text{choose}(52,2) = 5.88\%.$$

(b) Imagine *ordering* your 2 cards. No matter what your 1st card is, there are 51 equally likely choices for your 2nd card, and 3 of them give you a pocket pair.  $3/51 = 5.88\%$ .]

$$\mu = E(X) = .0588.$$

$$SD = \sigma = \sqrt{(.0588 * 0.9412)} = 0.235.$$

## Binomial Random Variables, ch. 5.2.

Suppose now  $X = \#$  of times something with prob.  $p$  occurs, out of  $n$  independent trials

Then  $X = \text{Binomial}(n, p)$ .

e.g. the number of pocket pairs, out of 10 hands.

Now  $X$  could  $= 0, 1, 2, 3, \dots$ , or  $n$ .

pmf:  $P(X = k) = \text{choose}(n, k) * p^k q^{n-k}$ .

e.g. say  $n=10, k=3$ :  $P(X = 3) = \text{choose}(10, 3) * p^3 q^7$ .

Why? Could have 1 1 1 0 0 0 0 0 0, or 1 0 1 1 0 0 0 0 0, etc.

$\text{choose}(10, 3)$  choices of places to put the 1's, and for each the prob. is  $p^3 q^7$ .

Key idea:  $X = Y_1 + Y_2 + \dots + Y_n$ , where the  $Y_i$  are independent and *Bernoulli* ( $p$ ).

If  $X$  is Bernoulli ( $p$ ), then  $\mu = p$ , and  $\sigma = \sqrt{pq}$ .

**If  $X$  is Binomial ( $n, p$ ), then  $\mu = np$ , and  $\sigma = \sqrt{npq}$ .**

#### 4. Binomial Random Variables, continued.

Suppose  $X$  = the number of pocket pairs you get in the next 100 hands.

What's  $P(X = 4)$ ? What's  $E(X)$ ?  $\sigma$ ?      $X = \text{Binomial}(100, 5.88\%)$ .

$$P(X = k) = \text{choose}(n, k) * p^k q^{n-k}.$$

So,  $P(X = 4) = \text{choose}(100, 4) * 0.0588^4 * 0.9412^{96} = 13.9\%$ , or 1 in **7.2**.

$$E(X) = np = 100 * 0.0588 = \mathbf{5.88}. \quad \sigma = \sqrt{100 * 0.0588 * 0.9412} = \mathbf{2.35}.$$

So, out of 100 hands, you'd *typically* get about 5.88 pocket pairs, +/- around 2.35.

## 5. Geometric random variables, ch 5.3.

Suppose now  $X = \#$  of trials until the first occurrence.

(Again, each trial is independent, and each time the probability of an occurrence is  $p$ .)

Then  $X = \text{Geometric}(p)$ .

e.g. the number of hands til you get your next pocket pair.

*[Including the hand where you get the pocket pair. If you get it right away, then  $X = 1$ .]*

Now  $X$  could be 1, 2, 3, ..., up to  $\infty$ .

pmf:  $P(X = k) = p^1 q^{k-1}$ .

e.g. say  $k=5$ :  $P(X = 5) = p^1 q^4$ . Why? Must be 0 0 0 0 1. Prob. =  $q * q * q * q * p$ .

**If  $X$  is Geometric ( $p$ ), then  $\mu = 1/p$ , and  $\sigma = (\sqrt{q}) \div p$ .**

e.g. Suppose  $X =$  the number of hands til your next pocket pair.  $P(X = 12)$ ?  $E(X)$ ?  $\sigma$ ?

$X = \text{Geometric}(5.88\%)$ .

$P(X = 12) = p^1 q^{11} = 0.0588 * 0.9412^{11} = \mathbf{3.02\%}$ .

$E(X) = 1/p = \mathbf{17.0}$ .  $\sigma = \text{sqrt}(0.9412) / 0.0588 = \mathbf{16.5}$ .

So, you'd typically *expect* it to take 17 hands til your next pair, +/- around 16.5 hands.

## 6. Negative binomial random variables, ch5.4.

Recall: if each trial is independent, and each time the probability of an occurrence is  $p$ , and  $X = \#$  of trials until the first occurrence, then:

$$X \text{ is Geometric } (p), \quad P(X = k) = p^1 q^{k-1}, \quad \mu = 1/p, \quad \sigma = (\sqrt{q}) \div p.$$

Suppose now  $X = \#$  of trials until the  $r$ th occurrence.

Then  $X = \text{negative binomial } (r, p)$ .

e.g. the number of hands you have to play til you've gotten  $r=3$  pocket pairs.

Now  $X$  could be 3, 4, 5, ..., up to  $\infty$ .

pmf:  $P(X = k) = \text{choose}(k-1, r-1) p^r q^{k-r}$ , for  $k = r, r+1, \dots$

e.g. say  $r=3$  &  $k=7$ :  $P(X = 7) = \text{choose}(6, 2) p^3 q^4$ .

Why? Out of the first 6 hands, there must be exactly  $r-1 = 2$  pairs. Then pair on 7th.

$P(\text{exactly 2 pairs on first 6 hands}) = \text{choose}(6, 2) p^2 q^4$ .  $P(\text{pair on 7th}) = p$ .

**If  $X$  is negative binomial  $(r, p)$ , then  $\mu = r/p$ , and  $\sigma = (\sqrt{rq}) \div p$ .**

e.g. Suppose  $X =$  the number of hands til your 12th pocket pair.  $P(X = 100)$ ?  $E(X)$ ?  $\sigma$ ?

$X = \text{Neg. binomial } (12, 5.88\%)$ .

$$P(X = 100) = \text{choose}(99, 11) p^{12} q^{88}$$

$$= \text{choose}(99, 11) * 0.0588^{12} * 0.9412^{88} = \mathbf{0.104\%}.$$

$$E(X) = r/p = 12/0.0588 \sim \mathbf{204}. \quad \sigma = \sqrt{12 * 0.9412} / 0.0588 = \mathbf{57.2}.$$

So, you'd typically *expect* it to take 204 hands til your 12th pair, +/- around 57.2 hands.

## 7. Moment generating functions, ch. 4.7

Suppose  $X$  is a random variable.  $E(X)$ ,  $E(X^2)$ ,  $E(X^3)$ , etc. are the *moments* of  $X$ .

$\phi_X(t) = E(e^{tX})$  is called the *moment generating function* of  $X$ .

Take derivatives with respect to  $t$  of  $\phi_X(t)$  and evaluate at  $t=0$  to get moments of  $X$ .

1<sup>st</sup> derivative  $(d/dt) e^{tX} = X e^{tX}$ ,  $(d/dt)^2 e^{tX} = X^2 e^{tX}$ , etc.

$(d/dt)^k E(e^{tX}) = E[(d/dt)^k e^{tX}] = E[X^k e^{tX}]$ , (see p.84)

so  $\phi'_X(0) = E[X^1 e^{0X}] = E(X)$ ,

$\phi''_X(0) = E[X^2 e^{0X}] = E(X^2)$ , etc.

**The moment gen. function  $\phi_X(t)$  uniquely characterizes the distribution of  $X$ .**

So to show that  $X$  is, say, Poisson, you just need to show that it has the moment generating function of a Poisson random variable.

**Also, if  $X_i$  are random variables with cdfs  $F_i$ , and  $\phi_{X_i}(t) \rightarrow \phi(t)$ , where  $\phi_X(t)$  is the moment generating function of  $X$  which has cdf  $F$ , then  $X_i \rightarrow X$  in distribution, i.e.**

**$F_i(y) \rightarrow F(y)$  for all  $y$  where  $F(y)$  is continuous, see p85.**

## Moment generating functions, continued.

$\phi_X(t) = E(e^{tX})$  is called the *moment generating function* of  $X$ .

Suppose  $X$  is Bernoulli (0.4). What is  $\phi_X(t)$ ?

$$E(e^{tX}) = (0.6) (e^{t(0)}) + (0.4) (e^{t(1)}) = 0.6 + 0.4 e^t.$$

Suppose  $X$  is Bernoulli (0.4) and  $Y$  is Bernoulli (0.7) and  $X$  and  $Y$  are independent.

What is the distribution of  $XY$ ?

$$\phi_{XY}(t) = E(e^{tXY}) = P(XY=0) (e^{t(0)}) + P(XY=1)(e^{t(1)})$$

$$= P(X=0 \text{ or } Y=0) (1) + P(X=1 \text{ and } Y=1)e^t$$

$$= [1 - P(X=1)P(Y=1)] + P(X=1)P(Y=1)e^t$$

$$= [1 - 0.4 \times 0.7] + 0.4 \times 0.7 e^t$$

$= 0.72 + 0.28e^t$ , which is the moment generating function of a Bernoulli (0.28) random variable. Therefore  $XY$  is Bernoulli (0.28).

What about  $Z = \min\{X, Y\}$ ?

$Z = XY$  in this case, since  $X$  and  $Y$  are 0 or 1, so the answer is the same.