Stat 100a: Introduction to Probability.

Outline for the day

1. A quick fact about normals.
2. Random walks.
3. Reflection principle.
4. Ballot theorem.
5. Avoiding zero.
6. Chip proportions and induction.
7. Doubling up.

The computer project is due on Sun Dec1 8:00pm.

HW3 is due Tue Dec3 and is on the course website.

http://www.stat.ucla.edu/~frederic/100a/F19 .

Thu Dec5 is the final exam, here in class, 11am to 12:15pm.

Again any notes and books are fine, and bring a pencil and a calculator. Also bring your student ID to the exam.

1. A fact about normals.

If X and Y are independent and both are normal, then $X+Y$ is normal, and so are $-X$ and $-Y$.

2. Random walks, ch. 7.6**.**

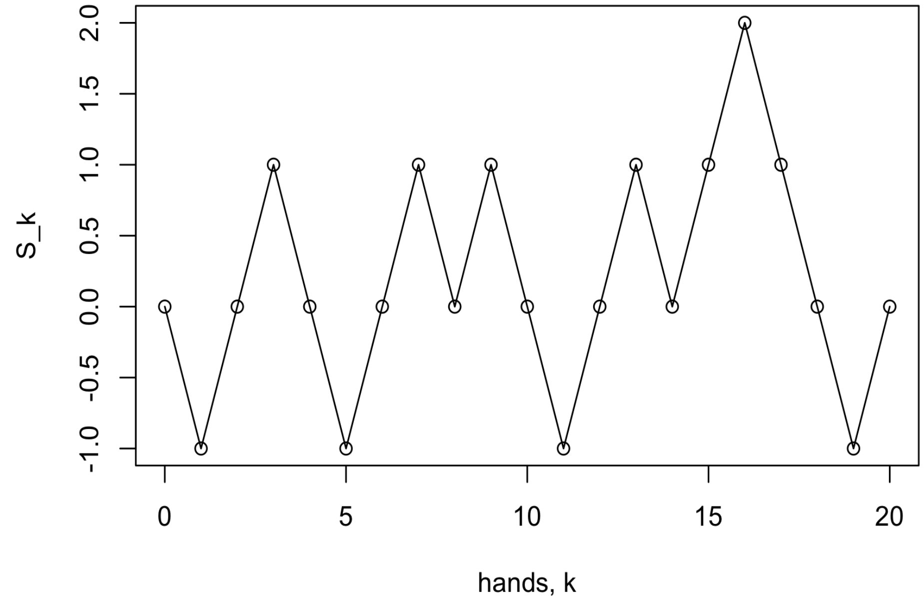
Suppose that $X_1, X_2, \ldots$, are iid,

and $S_k = X_0 + X_1 + \ldots + X_k$ for $k = 0, 1, 2, \ldots$.

The totals $\{S_0, S_1, S_2, \ldots\}$ form a *random walk*.

The classical *(simple)* case is when each X_i is

1 or -1 with probability ½ each.



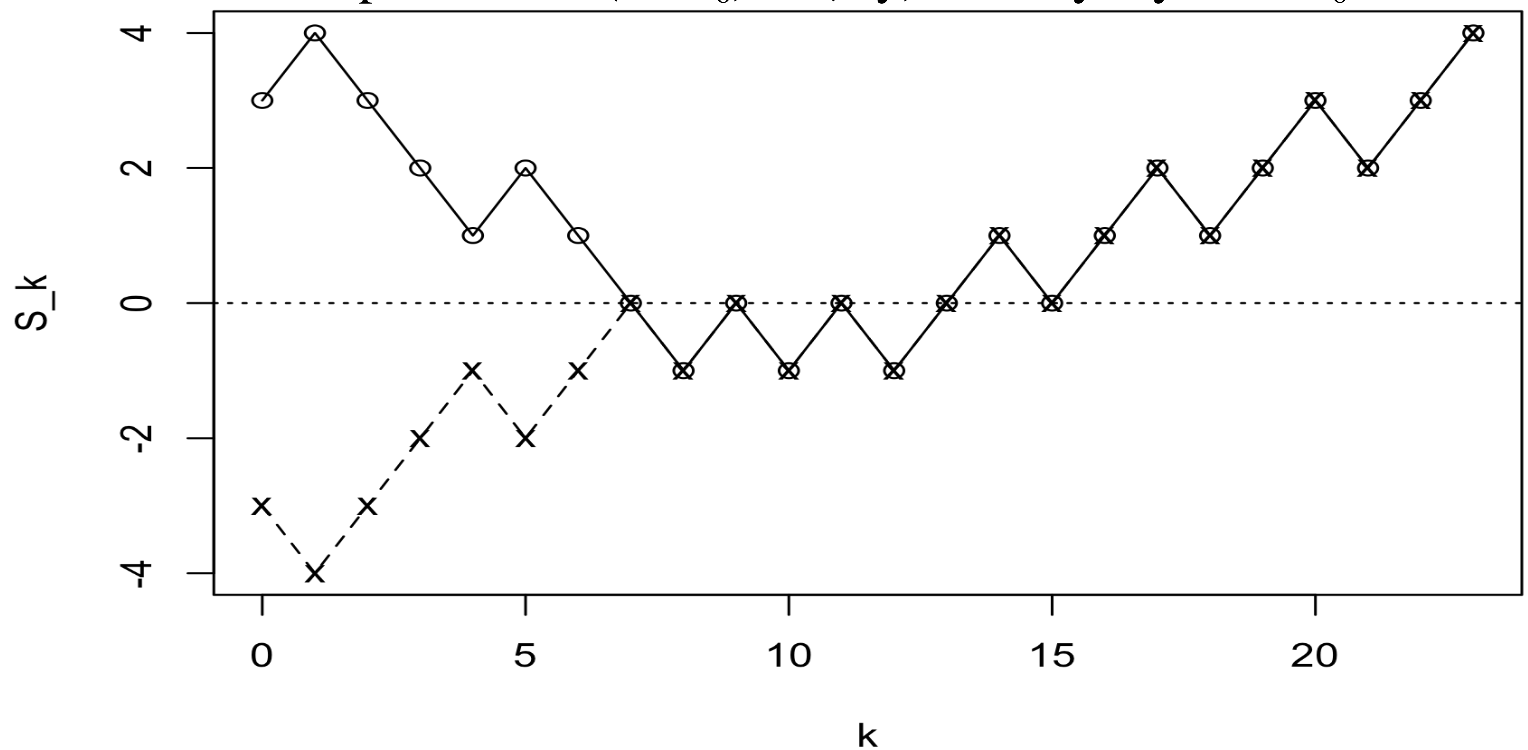
* *Reflection principle:* The number of paths from $(0,X_0)$ to (n,y) that touch the x-axis = the number of paths from $(0,-X_0)$ to (n,y), for any n,y, and $X_0 > 0$.

* *Ballot theorem:* In $n = a+b$ hands, if player A won a hands and B won b hands, where $a>b$, and if the hands are aired in random order, P(A won more hands than B *throughout* the telecast) $= (a-b)/n$.

[In an election, if candidate X gets x votes, and candidate Y gets y votes, where $x > y$, then the probability that X always leads Y throughout the counting is $(x-y) / (x+y)$.]

* For a simple random walk, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$, for any even n.

3. Reflection Principle. The number of paths from $(0,X_0)$ to (n,y) that touch the x-axis = the number of paths from $(0,-X_0)$ to (n,y), for any n,y, and $X_0 > 0$.



For each path from $(0,X_0)$ to (n,y) that touches the x-axis, you can reflect the first part til it touches the x-axis, to find a path from $(0,-X_0)$ to (n,y), and vice versa.

Total number of paths from $(0,-X_0)$ to (n,y) is easy to count: it's just C(n,a), where you go up *a* times and down *b* times.

[For example, to go from (0,-10) to (100, 20), you have to "profit" 30, so you go up a=65 times and down b=35 times, and the number of paths is C(100,65).

In general, $a-b = y - (-X_0) = y + X_0$. $a+b=n$, so $b = n-a$, $2a-n=y+X_0$, $a=(n+y+X_0)/2$].

4. Ballot theorem. In $n = a+b$ hands, if player A won a hands and B won b hands,

where $a>b$, and if the hands are aired in random order,

then P(A won more hands than B *throughout* the telecast) = $(a-b)/n$.

Proof: We know that, after $n = a+b$ hands, the total difference in hands won is $a-b$.

Let $x = a-b$.

We want to count the number of paths from $(1,1)$ to (n,x) that do not touch the x-axis.

By the reflection principle, the number of paths from $(1,1)$ to (n,x) that **do** touch the x-axis equals the total number of paths from $(1,-1)$ to (n,x).

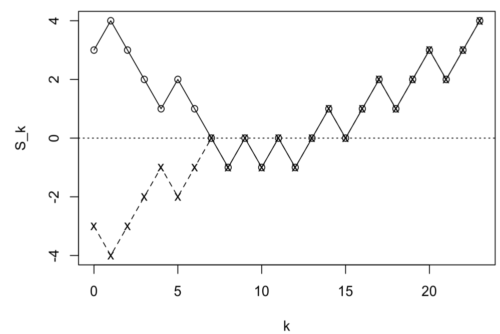
So the number of paths from $(1,1)$ to (n,x) that **do not** touch the x-axis equals the number of paths from $(1,1)$ to (n,x) minus the number of paths from $(1,-1)$ to (n,x)

$= C(n-1,a-1) – C(n-1,a)$

$= (n-1)! / [(a-1)! (n-a)!] – (n-1)! / [a! (n-a-1)!]$

$= \{n! / [a! (n-a)!]\} [(a/n) - (n-a)/n]$

$= C(n,a) (a-b)/n.$



And each path is equally likely, and has probability $1/C(n,a)$.

So, P(going from $(0,0)$ to (n,x) without touching the x-axis = $(a-b)/n$.

5. Avoiding zero.

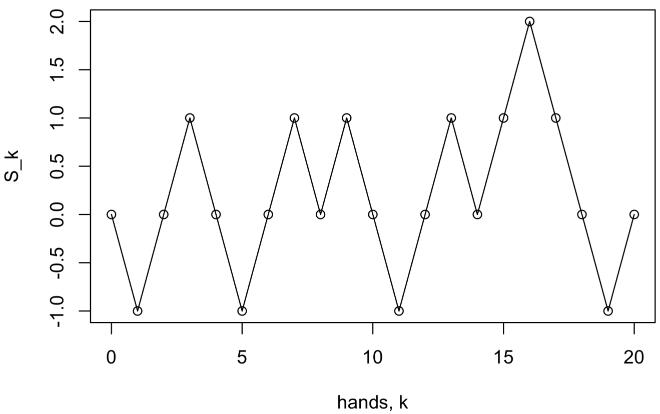
For a simple random walk, for any even # n, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.

Proof. The number of paths from (0,0) to (n, j) that don't touch the x-axis at positive times

= the number of paths from (1,1) to (n,j) that don't touch the x-axis at positive times

= paths from (1,1) to (n,j) - paths from (1,-1) to (n,j) by the *reflection principle*

$= N_{n-1,j-1} - N_{n-1,j+1}$.



Let $Q_{n,j} = P(S_n = j)$. By the logic above,

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = j) = ½[Q_{n-1,j-1} - Q_{n-1,j+1}]$.

Summing from j = 2 to ∞,

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n > 0)$

$= ½[Q_{n-1,1} - Q_{n-1,3}] + ½[Q_{n-1,3} - Q_{n-1,5}] + ½[Q_{n-1,5} - Q_{n-1,7}] + \ldots$ and these terms are eventually 0

$= (1/2)\, Q_{n-1,1}$.

Now note that $Q_{n-1,1} = P(S_n = 0)$, because to end up at (n, 0), you have to be at (n-1,1) and then go down, or at (n-1,-1) and then go up. So $P(S_n = 0) = (1/2)\, Q_{n-1,1} + (1/2)\, Q_{n-1,-1} = Q_{n-1,1}$.

Thus $P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n > 0) = ½\, P(S_n = 0)$. By the same arguments,

$P(S_1 < 0, S_2 < 0, \ldots, S_{n-1} < 0, S_n < 0) = 1/2\, P(S_n = 0)$.

So, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.

6. Chip proportions and induction, Theorem 7.6.6.

P(win a tournament) is proportional to your number of chips.

Simplified scenario. Suppose you either go up or down 1 each hand, with prob. 1/2.

Suppose there are n chips, and you have k of them.

Let p_k = P(win tournament given k chips) = P(random walk goes k -> n before hitting 0).

Now, clearly $p_0 = 0$. Consider p_1. From 1, you will either go to 0 or 2.

So, $p_1 = 1/2\ p_0 + 1/2\ p_2 = 1/2\ p_2$. That is, $p_2 = 2\ p_1$.

We have shown that $p_j = j\ p_1$, for j = 0, 1, and 2.

***(induction:)* Suppose that, for j = 0, 1, 2, …, m, $p_j = j\ p_1$.**

We will show that $p_{m+1} = (m+1)\ p_1$.

Therefore, $p_j = j\ p_1$ for all j.

That is, P(win the tournament) is prop. to your number of chips.

$p_m = 1/2\ p_{m-1} + 1/2\ p_{m+1}$. If $p_j = j\ p_1$ for $j \leq m$, then we have

$mp_1 = 1/2\ (m-1)p_1 + 1/2\ p_{m+1}$,

so $p_{m+1} = 2mp_1 - (m-1)\ p_1 = (m+1)p_1$.

7. Doubling up. Again, P(winning) = your proportion of chips.
Theorem 7.6.7, p152, describes another simplified scenario.
Suppose you either double each hand you play, or go to zero, each with probability 1/2.

Again, P(win a tournament) is prop. to your number of chips.

Again, $p_0 = 0$, and $p_1 = 1/2\ p_2 = 1/2\ p_2$, so again, $p_2 = 2\ p_1$.

We have shown that, for j = 0, 1, and 2, $p_j = j\ p_1$.

(induction:) **Suppose that, for $j \leq m$, $p_j = j\ p_1$.**

We will show that $p_{2m} = (2m)\ p_1$.

Therefore, $p_j = j\ p_1$ for all $j = 2^k$. That is, P(win the tournament) is prop. to # of chips.

This time, $p_m = 1/2\ p_0 + 1/2\ p_{2m}$. If $p_j = j\ p_1$ for $j \leq m$, then we have

$mp_1 = 0 + 1/2\ p_{2m}$, so $p_{2m} = 2mp_1$. Done.

In Theorem 7.6.8, p152, you have k of the n chips in play. Each hand, you gain 1 with prob. p, or lose 1 with prob. q=1-p.

Suppose $0<p<1$ and $p\neq 0.5$. Let $r = q/p$. Then P(you win the tournament) = $\mathbf{(1-r^k)/(1-r^n)}$.

The proof is again by induction, and is similar to the proof we did of Theorem 7.6.6.

8. Examples.

(Chen and Ankenman, 2006). Suppose that a \$100 winner-take-all tournament has $1024 = 2^{10}$ players. So, you need to double up 10 times to win. Winner gets \$102,400.

Suppose you have probability $p = 0.54$ to double up, instead of 0.5.

What is your expected profit in the tournament? (Assume only doubling up.)

Answer. P(winning) = 0.54^{10}, so exp. return = 0.54^{10} (\$102,400) = \$215.89. So exp. <u>profit</u> = \$115.89.

What if each player starts with 10 chips, and you gain a chip with $p = 54\%$ and lose a chip with $p = 46\%$? What is your expected profit?

Answer. $r = q/p = .46/.54 = .852$. P(you win) = $(1\text{-}r^{10})/(1\text{-}r^{10240})$ = 79.9%.

So exp. profit = .799(\$102400) - \$100 ~ \$81700.

Random Walk example.

Suppose you start with 1 chip at time 0 and that your tournament is like a simple random walk, but if you hit 0 you are done. P(you have not hit zero by time 47)?

We know that starting at 0, $P(Y_1 \neq 0, Y_2 \neq 0, \ldots, Y_{2n} \neq 0) = P(Y_{2n} = 0)$.

So, for a random walk starting at (0,0),

by symmetry $P(Y_1 > 0, Y_2 > 0, \ldots, Y_{48} > 0) = ½\ P(Y_1 \neq 0, Y_2 \neq 0, \ldots, Y_{2n} \neq 0)$

$= ½\ P(Y_{48} = 0) = ½$ Choose(48,24)(½)^{48}.

Also $P(Y_1 > 0, Y_2 > 0, \ldots, Y_{48} > 0) = P(Y_1 = 1, Y_2 > 0, \ldots, Y_{48} > 0)$

= P(start at 0 and win your first hand, and then stay above 0 for at least 47 more hands)

= P(start at 0 and win your first hand) x P(from (1,1), stay above 0 for ≥ 47 more hands)

= 1/2 P(starting with 1 chip, stay above 0 for at least 47 more hands).

So, multiplying both sides by 2,

P(starting with 1 chip, stay above 0 for at least 47 hands) = Choose(48,24)(½)^{48}

= 11.46%.