**Stat 100a, Introduction to Probability.**

Outline for the day:

1. Bayes's rule.
2. Random variables.
3. cdf, pmf, and density.
4. Greenstein and Farha.
5. Expected value.

HW1 is due Thu Oct14 in the first 5 min of class.

Exam1 is Tue Oct26.

Read through chapter 5. ♠ ♣ ♥ ♦

1. Bayes's rule.

Suppose that $B_1, B_2, B_n$ are disjoint events and that exactly one of them must occur.

Suppose you want $P(B_1 \mid A)$, but you only know $P(A \mid B_1)$, $P(A \mid B_2)$, etc.,

and you also know $P(B_1), P(B_2), \ldots, P(B_n)$.

Bayes' Rule: If $B_1, \ldots, B_n$ are disjoint events with $P(B_1 \text{ or } \ldots \text{ or } B_n) = 1$, then

$$P(B_i \mid A) = P(A \mid B_i) * P(B_i) \div [\sum P(A \mid B_j) P(B_j)].$$

Why? Recall: $P(X \mid Y) = P(X \& Y) \div P(Y)$. So $P(X \& Y) = P(X \mid Y) * P(Y)$.

$$P(B_1 \mid A) = P(A \& B_1) \div P(A)$$

$$= P(A \& B_1) \div [P(A \& B_1) + P(A \& B_2) + \ldots + P(A \& B_n)]$$

$$= P(A \mid B_1) * P(B_1) \div [P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \ldots + P(A \mid B_n)P(B_n)].$$

**Bayes's rule, continued.**

Bayes's rule: If $B_1, \dots, B_n$ are disjoint events with $P(B_1 \text{ or } \dots \text{ or } B_n) = 1$, then

$P(B_i \mid A) = P(A \mid B_i) * P(B_i) \div [\sum P(A \mid B_j)P(B_j)]$.

See example 3.4.1, p50. If a test is 95% accurate and 1% of the pop. has a condition, then given a random person from the population,

P(she has the condition | she tests positive)

= P(cond | +)

= P(+ | cond) P(cond) ÷ [P(+ | cond) P(cond) + P(+ | no cond) P(no cond)]

= 95% x 1% ÷ [95% x 1% + 5% x 99%]

~ 16.1%.

Tests for rare conditions must be extremely accurate.

**Bayes' rule example.**

Suppose P(your opponent has the nuts) = 1%, and P(opponent has a weak hand) = 10%.

Your opponent makes a huge bet. Suppose she'd only do that with the nuts or a weak hand, and that P(huge bet | nuts) = 100%, and P(huge bet | weak hand) = 30%.

What is P(nuts | huge bet)?

P(nuts | huge bet) =

$$\frac{\text{P(huge bet | nuts) * P(nuts)}}{\text{P(huge bet | nuts) P(nuts) + P(huge bet | horrible hand) P(horrible hand)}}$$

$$= \frac{100\% * 1\%}{100\% * 1\% + 30\% * 10\%}$$

= **25%.**

## 2. Random variables.

A *variable* is something that can take different numeric values.

A *random variable* (X) can take different numeric values with different probabilities.

X is *discrete* if all its possible values can be listed. If X can take any value in an interval like say [0,1], then X is *continuous*.

Ex. Two cards are dealt to you. Let X be 1 if you get a pair, and X is 0 otherwise.

P(X is 1) = 3/51 ~ 5.9%.    P(X is 0) ~ 94.1%.

Ex. A coin is flipped, and X=20 if heads, X=10 if tails.

The *distribution* of X means all the information about all the possible values X can take, along with their probabilities.

3. cdf, pmf, and density (pdf).

Any random variable has a *cumulative distribution function* (cdf):

$F(b) = P(X \le b)$.

If X is discrete, then it has a *probability mass function* (pmf):

$f(b) = P(X = b)$.

Continuous random variables are often characterized by their *probability density functions* (pdf, or *density*):

a function f(x) such that $P(X \text{ is in } B) = \int_B f(x)\, dx$.

4. Greenstein and Farha. 5. Expected value.

For a discrete random variable X with pmf f(b), the *expected value* of X = $\sum$ b f(b).

The sum is over all possible values of b.   (continuous random variables later…)

The expected value is also called the *mean* and denoted E(X) or $\mu$.

Ex: 2 cards are dealt to you. X = 1 if pair, 0 otherwise.

P(X is 1) ~ 5.9%, P(X is 0) ~ 94.1%.

E(X) = (1 x 5.9%) + (0 x 94.1%) = 5.9%, or 0.059.

Ex. Coin, X=20 if heads, X=10 if tails.

E(X) = (20x50%) + (10x50%) = 15.

Ex. Lotto ticket. f($10million) = 1/choose(52,6) = 1/20million, f($0) = 1-1/20mil.

E(X) = ($10mil x 1/20million) = $0.50.

The expected value of X represents a *best guess* of X.

Compare with the *sample mean*, $\overline{X} = (X_1 + X_1 + \ldots + X_n) / n$.

Some reasons why Expected Value applies to poker:

- Tournaments: some game theory results suggest that, in symmetric, winner-take-all games, the optimal strategy is the one which uses the *myopic rule*: that is, given any choice of options, always choose the one that maximizes your *expected value*.
- Laws of large numbers: Some statistical theory indicates that, if you repeat an experiment over and over repeatedly, your long-term average will ultimately converge to the *expected value*. So again, it makes sense to try to maximize expected value when playing poker (or making deals).
- Checking results: A great way to check whether you are a long-term winning or losing player, or to verify if a certain strategy works or not, is to check whether the sample mean is positive and to see if it has converged to the *expected value*.

Heads up with AA.

Dan Harrington says that, "with a hand like AA, you really want to be one-on-one." True or false?

* Best possible pre-flop situation is to be all in with AA vs A8, where the 8 is the same suit as one of your aces, in which case you're about 94% to win. (the 8 could equivalently be a 6,7, or 9.) If you are all in for $100, then your expected holdings afterwards are $188.

a) In a more typical situation: you have AA against TT. You're 80% to win, so your expected value is $160.

b) Suppose that, after the hand vs TT, you get QQ and get up against someone with A9 who has more chips than you do. The chance of you winning this hand is 72%, and the chance of you winning both this hand and the hand above is 58%, so your expected holdings after both hands are $232:
you have 58% chance of having $400, and 42% chance to have $0.

c) Now suppose instead that you have AA and are all in against 3 callers with A8, KJ suited, and 44. Now you're 58.4% to quadruple up. So your expected holdings after the hand are $234, and the situation is just like (actually slightly better than) #1 and #2 combined: 58.4% chance to hold $400, and 41.6% chance for $0.

* So, being all-in with AA against 3 players is much better than being all-in with AA against one player: in fact, it's about like having two of these lucky one-on-one situations.

What about if you have 55?

a) You have $100 and 55 and are up against A9. You are 56% to win, so your expected value is $112.

b) You have $100 and 55 and are up against A9, KJ, and QJs. Seems pretty terrible, doesn't it? But you have a probability of 27.3% to quadruple, so your expected value is

0.273 x $400 = $109. About the same as #1!

[ For these probabilities, see the online Texas Holdem odds calculator at http://www.cardplayer.com .]

Expected value and pot odds.

Suppose someone bets (or raises) you, going all-in. What should your chances of winning be in order for you to correctly call?

Let B = the amount bet to you, i.e. the additional amount you'd need to put in if you want to call. So, if you bet 100 & your opponent with 800 left went all-in, B = 700.

Let POT = the amount in the pot right now (including your opponent's bet).

Let p = your probability of winning the hand if you call. So prob. of losing = 1-p.

Let CHIPS = the number of chips you have right now.

If you call, then E[your chips at end] = (CHIPS - B)(1-p) + (CHIPS + POT)(p)

= CHIPS(1-p+p) - B(1-p) + POT(p) = CHIPS - B + Bp + POTp

If you fold, then E[your chips at end] = CHIPS.

You want your expected number of chips to be maximized, so it's worth calling if

-B + Bp + POTp > 0, i.e. if **p > B / (B+POT).**

From previous slide, to call an all-in, need $P(win) > B \div (B+pot)$.
Expressed as an *odds ratio*, this is sometimes referred to as *pot odds* or *express odds*.

If the bet is not all-in & another betting round is still to come, need

$$P(win) > wager \div (wager + winnings),$$

where winnings = pot + amount you'll win on later betting rounds,
wager = total amount you will wager including the current round & later rounds, assuming no folding.

The terms *Implied-odds* / *Reverse-implied-odds* describe the cases where *winnings > pot* or where *wager > B*, respectively. See p66.

You will not be tested on implied or reverse implied odds in this course.

Example: 2006 World Series of Poker (WSOP). ♠ ♣ ♥ ♦

Blinds: 200,000/400,000, + 50,000 ante.

Jamie Gold (4♠ 3♣): 60 million chips. Calls.

Paul Wasicka (8♠ 7♠): 18 million chips. Calls.

Michael Binger (A♦ 10♦): 11 million chips. Raises to $1,500,000.

Gold & Wasicka call. (pot = 4,650,000) Flop: 6♠ 10♣ 5♠.

- Wasicka checks, Binger bets $3,500,000. (pot = 8,150,000)
- Gold moves all-in for 16,450,000. (pot = 24,600,000)
- Wasicka folds. Q: Based on expected value, should he have called?

If Binger will fold, then Wasicka's chances to beat Gold must be at least 16,450,000 / (24,600,000 + 16,450,000) = 40.1%.

If Binger calls, it's a bit complicated, but basically Wasicka's chances must be at least 16,450,000 / (24,600,000 + 16,450,000 + 5,950,000) = 35.0%.