Stat 100a: Introduction to Probability.

<u>Outline for the day</u>

1. Return midterms.
2. Conditional expectation.
3. Law of Large Numbers, LLN.
4. Bivariate and marginal density.
5. CLT.
6. CIs.
7. Sample size calculations.

PLEASE BE SILENT UNTIL I AM FINISHED RETURNING ALL THE MIDTERMS!

Everyone's score is boosted by 1 point out of 14.
So if it says 9 in red, circled, on the last problem of your exam, then you effectively got a score of 10/14.

HW3 is due Tue Nov 30 by email to frederic@stat.ucla.edu .

Tue Nov 23, class will be over zoom and no office hour that day.
Meeting ID: 915 0941 1456. Password: 235711

Conditional expectation, E(Y | X), ch. 7.2.

Suppose X and Y are discrete.

Then E(Y | X=j) is defined as $\sum_k$ k P(Y = k | X = j), just as you'd think.

E(Y | X) is a **random variable** such that E(Y | X) = E(Y | X=j) whenever X = j.

For example, let X = the # of spades in your hand, and Y = the # of clubs in your hand.

a) What's E(Y)? **b)** What's E(Y|X)? **c)** What's P(E(Y|X) = 1/3)?

a. E(Y) = 0P(Y=0) + 1P(Y=1) + 2P(Y=2)

= 0 + 13x39/C(52,2) + 2 C(13,2)/C(52,2) = 0.5.

b. X is either 0, 1, or 2. If X = 0, then E(Y|X) = E(Y | X=0) and

E(Y | X=0) = 0 P(Y=0 | X=0) + 1 P(Y=1 | X=0) + 2 P(Y=2 | X = 0)

= 0 + 13x26/C(39,2) + 2 C(13,2) / C(39,2) = **2/3**.

E(Y | X=1) = 0 P(Y=0 | X=1) + 1 P(Y=1 | X=1) + 2 P(Y=2 | X = 1)

= 0 + 13/39 + 2 (0) = **1/3**.

E(Y | X=2) = 0 P(Y=0 | X=2) + 1 P(Y=1 | X=2) + 2 P(Y=2 | X = 2)

= 0 + 1 (0) + 2(0) = **0**.

So E(Y | X = 0) = 2/3, E(Y | X = 1) = 1/3, and E(Y | X = 2) = 0. That's what E(Y|X) is

c. P(E(Y|X) = 1/3) is just P(X=1) = 13x39/C(52,2) ~ 38.24%.

Law of Large Numbers (LLN) and the Fundamental Theorem of Poker, ch 7.3.

David Sklansky, *The Theory of Poker,* 1987.

> "Every time you play a hand differently from the way you would have played it if you could see all your opponents' cards, they gain; and every time you play your hand the same way you would have played it if you could see all their cards, they lose. Conversely, every time opponents play their hands differently from the way they would have if they could see all your cards, you gain; and every time they play their hands the same way they would have played if they could see all your cards, you lose."

Meaning?

LLN: If X_1, X_2 , etc. are iid with expected value μ and sd σ, then $\overline{X}_n$ ---> μ.

Any short term good or bad luck will ultimately become *negligible* to the sample mean.

However, this does not mean that good luck and bad luck will ultimately cancel out. See p132.

Bivariate and marginal density.

Suppose X and Y are random variables.

If X and Y are discrete, we can define the joint pmf f(x,y) = P(X = x and Y = y).

Suppose X and Y are continuous for the rest of this page.

Define the bivariate or joint pdf f(x,y) as a function with the properties that f(x,y) ≥ 0, and for any a,b,c,d,

$$P(a \le X \le b \text{ and } c \le Y \le d) = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

The integral $\int_{-\infty}^{\infty} f(x,y)\, dy = f(x)$, the pdf of X, and this function f(x) is sometimes called the *marginal* density of X. Similarly $\int_{-\infty}^{\infty} f(x,y)\, dx$ is the marginal pdf of Y.

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} x\, [\int_{-\infty}^{\infty} f(x,y)\, dy]\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx.$

Just as P(A|B) = P(AB)/P(B), f(x|y) = f(x,y)/f(y).

X and Y are independent iff. $f(x,y) = f_x(x)f_y(y)$.

Now $E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dy\, dx$. This can be useful to find cov(X,Y) = E(XY) – E(X)E(Y).

What is $E(X^2Y+e^Y)$? It $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2y+e^y)\, f(x,y)\, dy\, dx.$

Bivariate and marginal density.

Suppose the joint density of X and Y is f(x,y) = a exp(x+y), for X and Y in (0,1) x (0,1). What is a? What is the marginal density of Y? What type of distribution does X have conditional on Y? What is E(X|Y)? What is the mean of X when Y = .2? Are X and Y independent?

$\int\int$a exp(x+y) dxdy = 1 = a$\int\int$exp(x)exp(y)dxdy = a$\int_0^1$ exp(x) dx $\int_0^1$ exp(y) dy = a$(e-1)^2$, so a = $(e-1)^{-2}$.

The marginal density of Y is f(y) = $\int_0^1$ a exp(x+y) dx = a exp(y) $\int_0^1$ exp(x)dx = a exp(y)(e-1) = exp(y)/(e-1).

Conditional on Y, the density of X is f(x|y) = f(x,y)/f(y) = a exp(x+y)(e-1)/exp(y) = exp(x)/(e-1). So X|Y is like an exponential(1) random variable restricted to (0,1).

E(X|Y) = $\int_0^1$ x exp(x)/(e-1) dx = 1/(e-1) [x exp(x) - $\int$exp(x)dx] = 1/(e-1) [x exp(x) – exp(x)$]_0^1$ = 1/(e-1) [e – e – 0 + 1] = 1/(e-1).
When Y = .2, E(X|Y) = 1/(e-1).

f(y) = exp(y)/(e-1) and similarly f(x) = exp(x)/(e-1),
so f(x)f(y) = exp(x+y)/$(e-1)^2$ = f(x,y). Therefore, X and Y are independent.

Central Limit Theorem (CLT), ch 7.4.

Sample mean $\overline{X}_n = \sum X_i / n$

iid: independent and identically distributed.

Suppose X_1, X_2, etc. are iid with expected value μ and sd σ,

LAW OF LARGE NUMBERS (LLN):

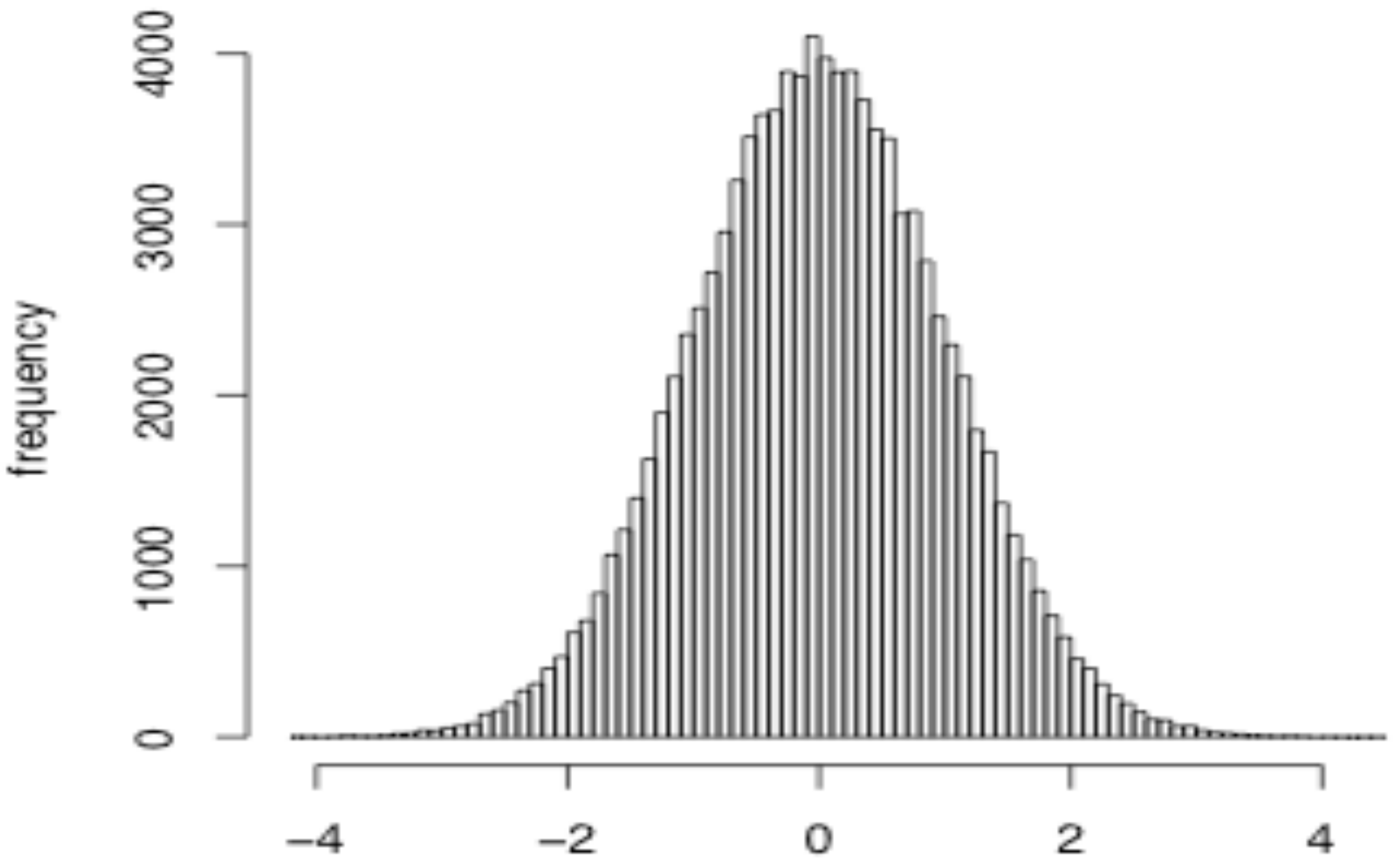
$\overline{X}_n \rightarrow \mu$.

CENTRAL LIMIT THEOREM (CLT):

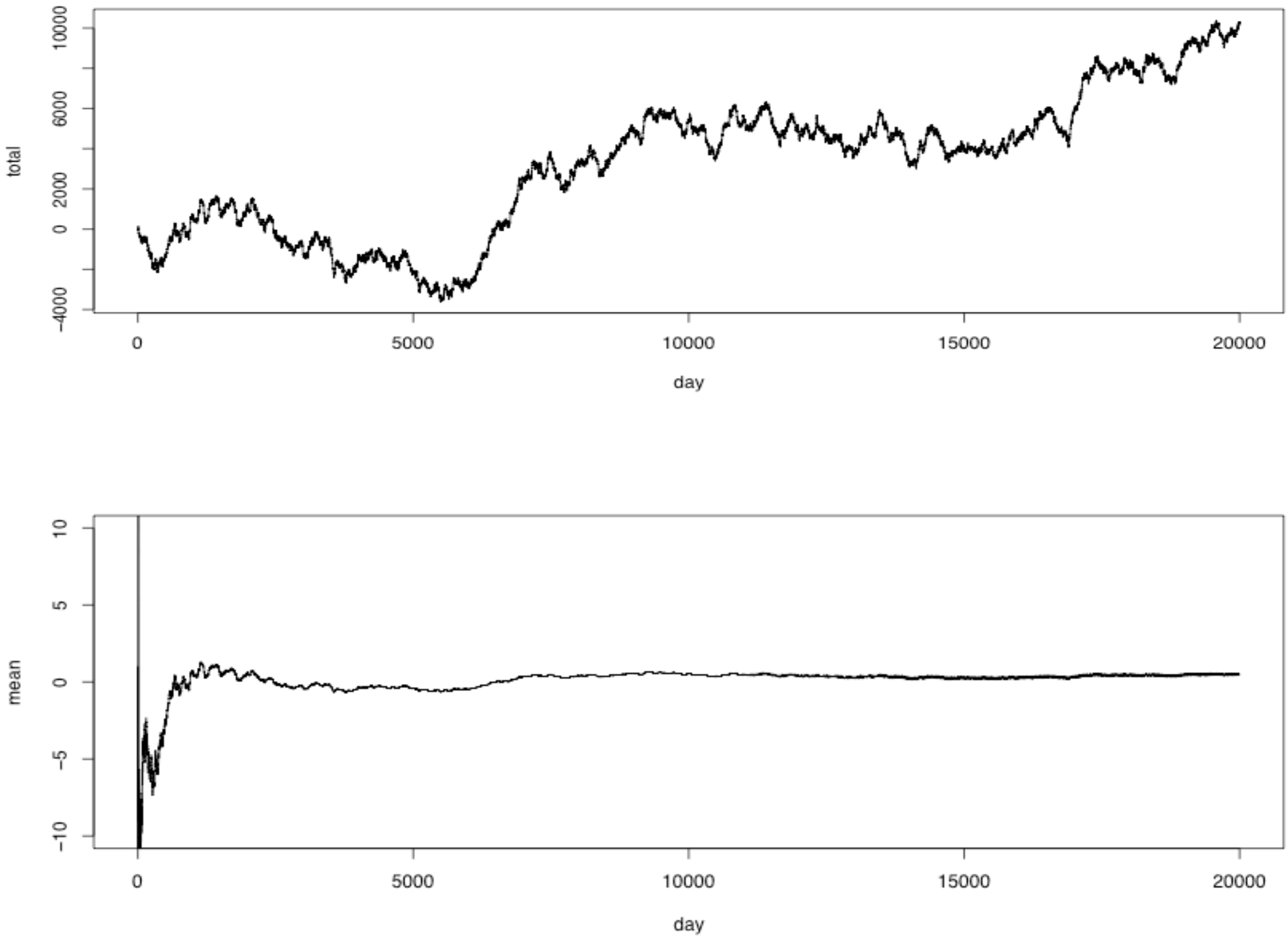
$(\overline{X}_n - \mu) \div (\sigma/\sqrt{n}) \rightarrow$ Standard Normal.

Useful for tracking results.

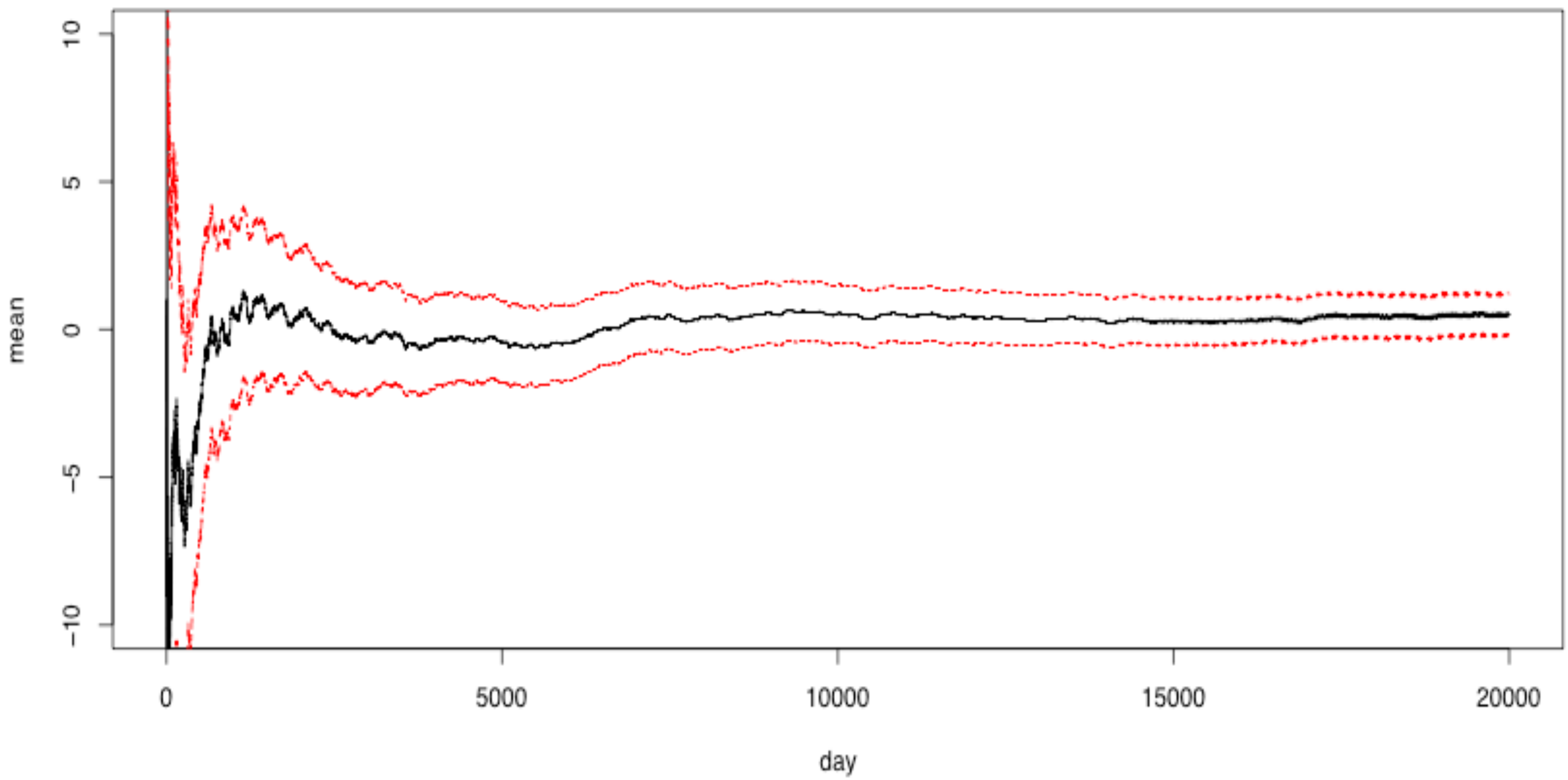
95% between -1.96 and 1.96



Truth: -49 to 51, exp. value $\mu = 1.0$



Truth: uniform on -49 to 51. $\mu = 1.0$
Estimated using $\overline{X}_n$ +/- 1.96 $\sigma/\sqrt{n}$
= .95 +/- 0.28 in this example



<u>Central Limit Theorem (CLT):</u> if $X_1, X_2 \ldots, X_n$ are iid with mean μ & SD σ, then

$$(\overline{X}_n - \mu) \div (\sigma/\sqrt{n}) \text{ ---> Standard Normal. (mean 0, SD 1).}$$

In other words, $\overline{X}_n$ has mean μ and a standard deviation of $\sigma \div \sqrt{n}$.

Two interesting things about this:

(i) As n --> ∞, $\overline{X}_n$ --> ***normal.*** Even if X_i are far from normal.

e.g. *average* number of pairs per hand, out of n hands. X_i are 0-1 (Bernoulli).

μ = p = P(pair) = 3/51 = 5.88%. σ = √(pq) = √(5.88% x 94.12%) = 23.525%.

(ii) We can use this to find **a range** where $\overline{X}_n$ is likely to be.

About 95% of the time, a std normal random variable is within -1.96 to +1.96.

So 95% of the time, $(\overline{X}_n - \mu) \div (\sigma/\sqrt{n})$ is within -1.96 to +1.96.

So 95% of the time, $(\overline{X}_n - \mu)$ is within $-1.96\,(\sigma/\sqrt{n})$ to $+1.96\,(\sigma/\sqrt{n})$.

So 95% of the time, $\overline{X}_n$ is within $\mu - 1.96\,(\sigma/\sqrt{n})$ to $\mu + 1.96\,(\sigma/\sqrt{n})$.

That is, 95% of the time, $\overline{X}_n$ is in the interval μ +/- 1.96 (σ/√n).

= 5.88% +/- 1.96(23.525%/√n). For n = 1000, this is 5.88% +/- 1.458%.

For n = 1,000,000 get 5.88% +/- 0.0461%.

Another CLT Example

Central Limit Theorem (CLT): if $X_1, X_2 \ldots, X_n$ are iid with mean μ & SD σ, then

$$(\overline{X_n} - \mu) \div (\sigma/\sqrt{n}) \dashrightarrow \text{Standard Normal. (mean 0, SD 1).}$$

In other words, $\overline{X_n}$ is like a draw from a normal distribution with mean μ and standard deviation of $\sigma \div \sqrt{n}$.

That is, 95% of the time, $\overline{X_n}$ is in the interval μ +/- 1.96 $(\sigma/\sqrt{n})$.

Q. Suppose you average \$5 profit per hour, with a SD of \$60 per hour. If you play 1600 hours, let Y be your average profit over those 1600 hours. Find a range where Y is 95% likely to fall.

A. We want μ +/- 1.96 $(\sigma/\sqrt{n})$, where μ = \$5, σ = \$60, and n=1600. So the answer is

\$5 +/- 1.96 x \$60 / √(1600)

= \$5 +/- \$2.94, or the range [\$2.06, \$7.94].

Confidence Intervals (CIs) for μ, ch 7.5.

<u>Central Limit Theorem (CLT):</u> if X_1 , X_2 …, X_n are iid with mean μ & SD σ, then

$$(\overline{X}_n - \mu) \div (\sigma/\sqrt{n}) \;\text{--->}\; \text{Standard Normal. (mean 0, SD 1).}$$

So, 95% of the time, $\overline{X}_n$ is in the interval μ +/- 1.96 $(\sigma/\sqrt{n})$.

Typically you know $\overline{X}_n$ but not μ. Turning the blue statement above around a bit means that 95% of the time, μ is in the interval $\overline{X}_n$ +/- 1.96 $(\sigma/\sqrt{n})$.

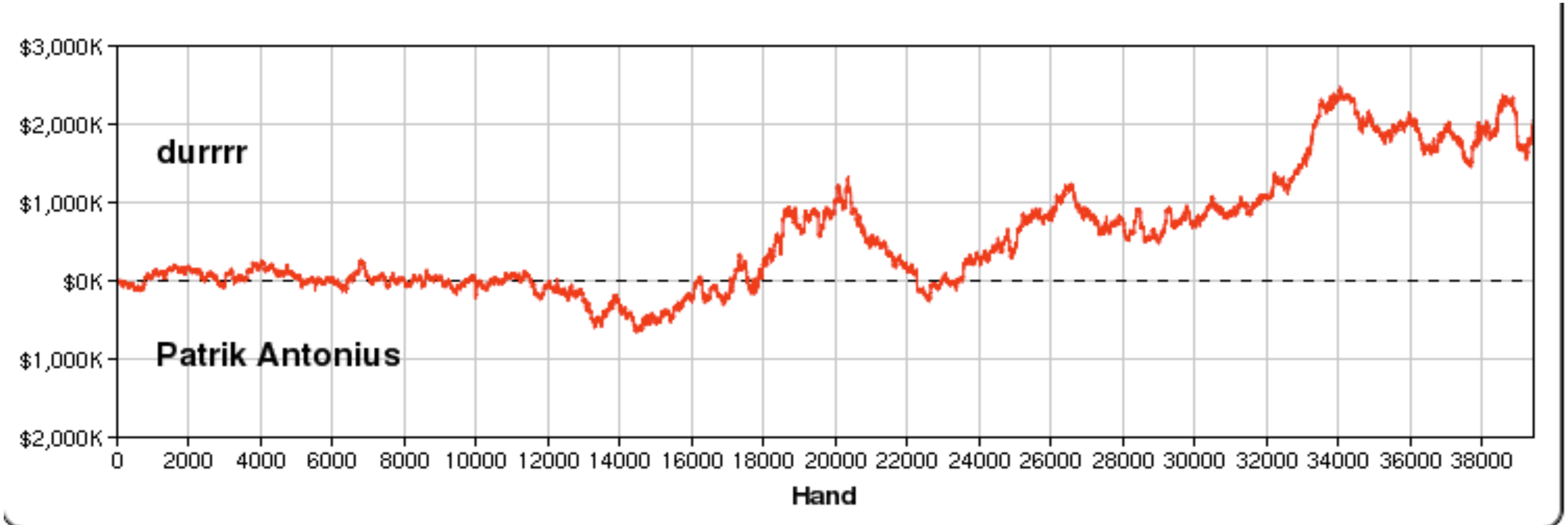
This range $\overline{X}_n$ +/- 1.96 $(\sigma/\sqrt{n})$ is called a 95% confidence interval (CI) for μ.

[Usually you don't know σ and have to estimate it using the sample std deviation, s, of your data, and $(\overline{X}_n - \mu) \div (s/\sqrt{n})$ has a t_{n-1} distribution if the X_i are normal. For n>30, t_{n-1} is so similar to normal though.]

1.96 $(\sigma/\sqrt{n})$ is called the *margin of error.*

The range $\overline{X}_n$ +/- 1.96 (σ/√n) is a 95% confidence interval for μ. 1.96 (σ/√n)

(from fulltiltpoker.com:)



Based on the data, can we conclude Dwan is a better player? Is his longterm avg. $\mu > 0$?

Over these 39,000 hands, Dwan profited $2 million. $51/hand. sd ~ $10,000.

95% CI for μ is $51 +/- 1.96 ($10,000 / √39,000) = $51 +/- $99 = (-$48, $150).

Results are inconclusive, even after 39,000 hands!

Sample size calculation. How many ***more*** hands are needed?

If Dwan keeps winning $51/hand, then we want n so that the margin of error = $51.

1.96 (σ/√n) = $51 means 1.96 ($10,000) / √n = $51, so n = [(1.96)($10,000)/($51)]^2 ~ 148,000, so about 109,000 *more* hands.