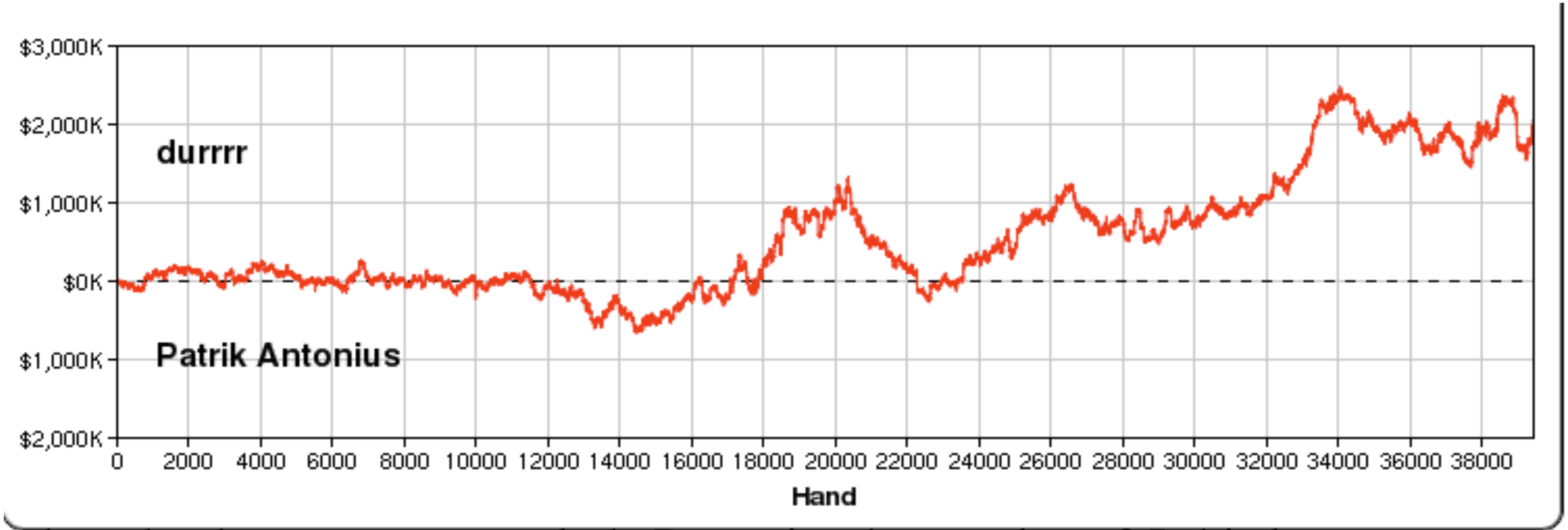
Stat 100a: Introduction to Probability.

Outline for the day

1. Sample size, and a quick fact about normals.
2. Random walks.
3. Reflection principle.
4. Ballot theorem.
5. Avoiding zero.
6. Chip proportions and induction.
7. Doubling up.

The range $\overline{X}_n$ +/- 1.96 (σ/√n) is a 95% confidence interval for μ. 1.96 (σ/√n)

(from fulltiltpoker.com:)



Based on the data, can we conclude Dwan is a better player? Is his longterm avg. $\mu > 0$?

Over these 39,000 hands, Dwan profited $2 million. $51/hand. sd ~ $10,000.

95% CI for μ is $51 +/- 1.96 ($10,000 / √39,000) = $51 +/- $99 = (-$48, $150).

Results are inconclusive, even after 39,000 hands!

Sample size calculation. How many *more* hands are needed?

If Dwan keeps winning $51/hand, then we want n so that the margin of error = $51.

1.96 (σ/√n) = $51 means 1.96 ($10,000) / √n = $51, so n = [(1.96)($10,000)/($51)]^2 ~ 148,000, so about 109,000 *more* hands.

A fact about normals.

If X and Y are independent and both are normal, then X+Y is normal, and so are –X and –Y.

2. Random walks, ch. 7.6**.**

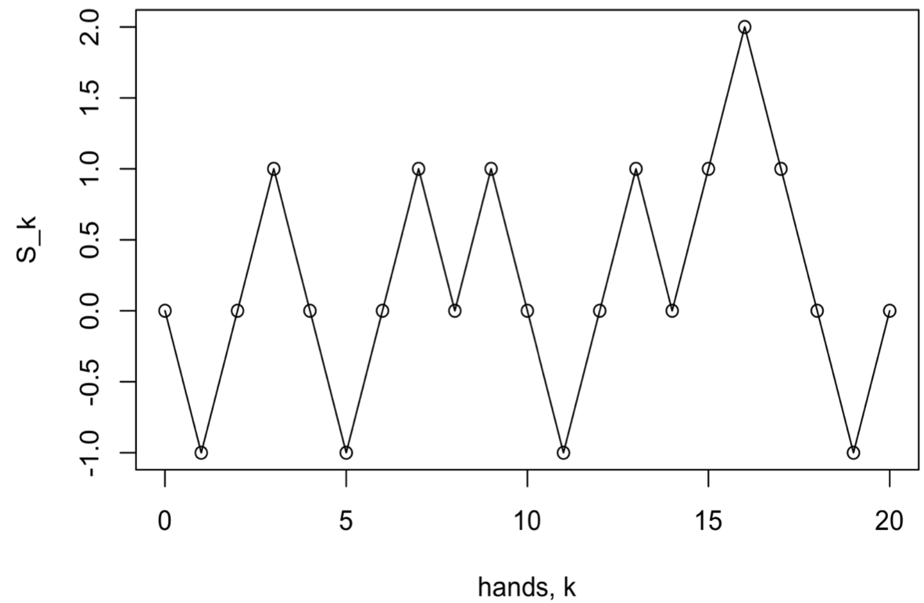
Suppose that $X_1, X_2, \ldots$, are iid,

and $S_k = X_0 + X_1 + \ldots + X_k$ for $k = 0, 1, 2, \ldots$.

The totals $\{S_0, S_1, S_2, \ldots\}$ form a <u>*random walk*</u>.

The classical *(simple)* case is when each X_i is

1 or -1 with probability ½ each.



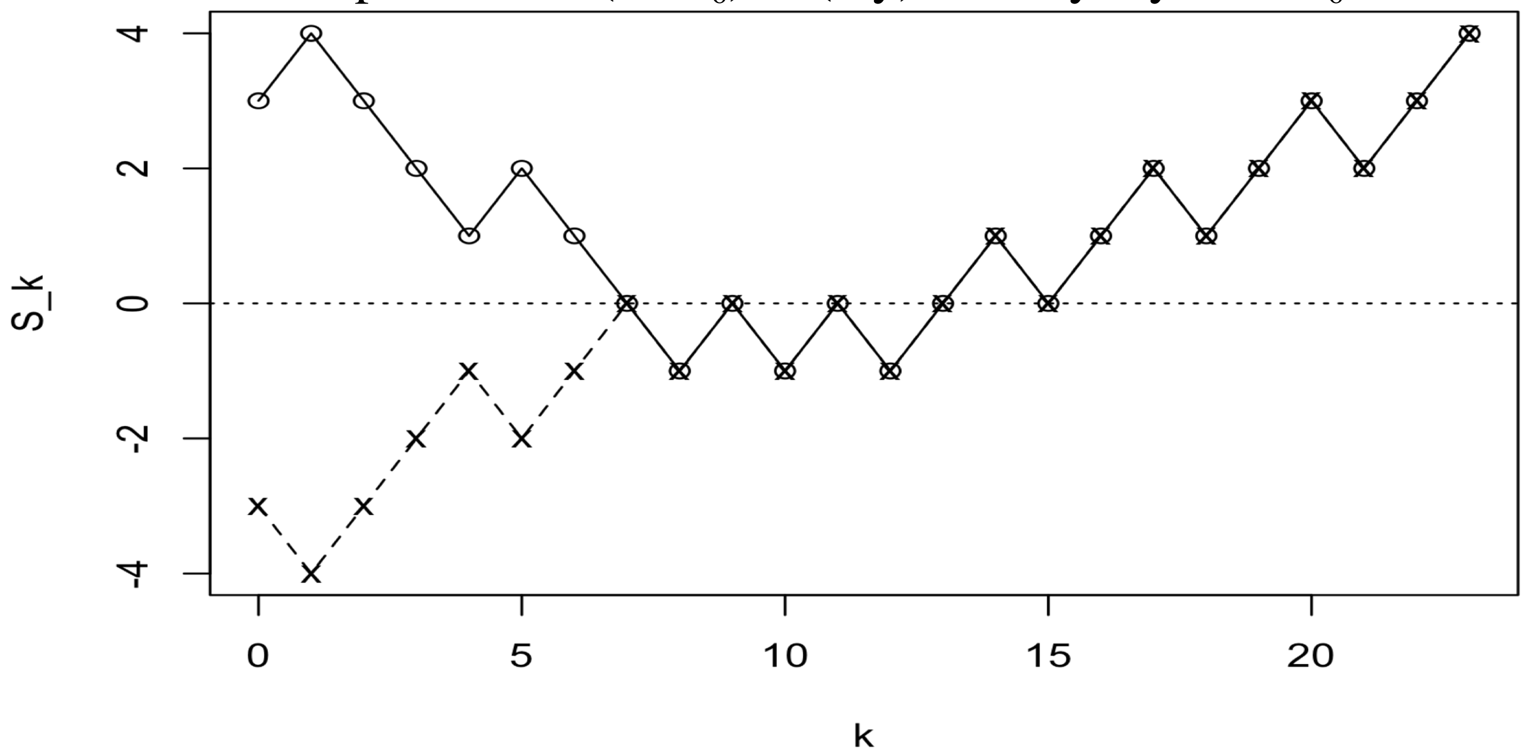
* <u>*Reflection principle*</u>*:* The number of paths from $(0,X_0)$ to (n,y) that touch the x-axis = the number of paths from $(0,-X_0)$ to (n,y), for any n,y, and $X_0 > 0$.

* <u>*Ballot theorem*</u>*:* In n = a+b hands, if player A won a hands and B won b hands, where a>b, and if the hands are aired in random order, P(A won more hands than B *throughout* the telecast) = (a-b)/n.

[In an election, if candidate X gets x votes, and candidate Y gets y votes, where x > y, then the probability that X always leads Y throughout the counting is (x-y) / (x+y).]

* For a simple random walk, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$, for any even n.

3. Reflection Principle. The number of paths from $(0,X_0)$ to (n,y) that touch the x-axis = the number of paths from $(0,-X_0)$ to (n,y), for any n,y, and $X_0 > 0$.



For each path from $(0,X_0)$ to (n,y) that touches the x-axis, you can reflect the first part til it touches the x-axis, to find a path from $(0,-X_0)$ to (n,y), and vice versa.

Total number of paths from $(0,-X_0)$ to (n,y) is easy to count: it's just C(n,a), where you go up *a* times and down *b* times.

[For example, to go from (0,-10) to (100, 20), you have to "profit" 30, so you go up a=65 times and down b=35 times, and the number of paths is C(100,65).

In general, $a-b = y - (-X_0) = y + X_0$. $a+b=n$, so $b = n-a$, $2a-n=y+X_0$, $a=(n+y+X_0)/2$].

4. Ballot theorem. In n = a+b hands, if player A won a hands and B won b hands, where a>b, and if the hands are aired in random order,

then P(A won more hands than B *throughout* the telecast) = (a-b)/n.

Proof: We know that, after n = a+b hands, the total difference in hands won is a-b.

Let x = a-b.

We want to count the number of paths from (1,1) to (n,x) that do not touch the x-axis.

By the reflection principle, the number of paths from (1,1) to (n,x) that **do** touch the x-axis equals the total number of paths from (1,-1) to (n,x).

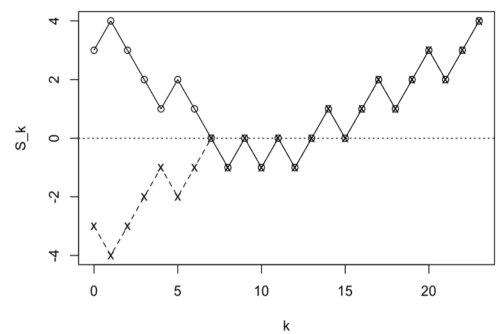
So the number of paths from (1,1) to (n,x) that **do not** touch the x-axis equals the number of paths from (1,1) to (n,x) minus the number of paths from (1,-1) to (n,x)

= C(n-1,a-1) – C(n-1,a)

= (n-1)! / [(a-1)! (n-a)!] – (n-1)! / [a! (n-a-1)!]

= {n! / [a! (n-a)!]} [(a/n) - (n-a)/n]

= C(n,a) (a-b)/n.



And each path is equally likely, and has probability 1/C(n,a).

So, P(going from (0,0) to (n,x) without touching the x-axis = (a-b)/n.

5. Avoiding zero.

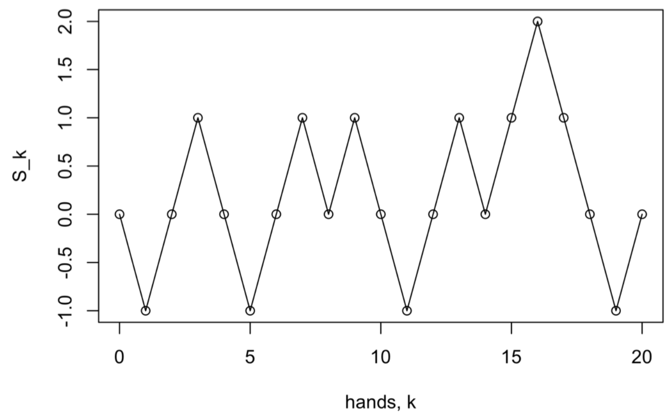
For a simple random walk, for any even # n, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.

Proof. The number of paths from (0,0) to (n, j) that don't touch the x-axis at positive times

= the number of paths from (1,1) to (n,j) that don't touch the x-axis at positive times

= paths from (1,1) to (n,j) - paths from (1,-1) to (n,j) by the *reflection principle*

$= N_{n-1,j-1} - N_{n-1,j+1}$.



Let $Q_{n,j} = P(S_n = j)$. By the logic above,

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = j) = ½[Q_{n-1,j-1} - Q_{n-1,j+1}]$.

Summing from j = 2 to ∞,

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n > 0)$

$= ½[Q_{n-1,1} - Q_{n-1,3}] + ½[Q_{n-1,3} - Q_{n-1,5}] + ½[Q_{n-1,5} - Q_{n-1,7}] + \ldots$ and these terms are eventually 0

$= (1/2)\, Q_{n-1,1}$.

Now note that $Q_{n-1,1} = P(S_n = 0)$, because to end up at (n, 0), you have to be at (n-1,1) and then go down, or at (n-1,-1) and then go up. So $P(S_n = 0) = (1/2)\, Q_{n-1,1} + (1/2)\, Q_{n-1,-1} = Q_{n-1,1}$.

Thus $P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n > 0) = ½\, P(S_n = 0)$. By the same arguments,

$P(S_1 < 0, S_2 < 0, \ldots, S_{n-1} < 0, S_n < 0) = 1/2\, P(S_n = 0)$.

So, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.

6. Chip proportions and induction, Theorem 7.6.6.

P(win a tournament) is proportional to your number of chips.

Simplified scenario. Suppose you either go up or down 1 each hand, with prob. 1/2.

Suppose there are n chips, and you have k of them.

Let p_k = P(win tournament given k chips) = P(random walk goes k -> n before hitting 0).

Now, clearly $p_0 = 0$. Consider p_1. From 1, you will either go to 0 or 2.

So, $p_1 = 1/2\ p_0 + 1/2\ p_2 = 1/2\ p_2$. That is, $p_2 = 2\ p_1$.

We have shown that $p_j = j\ p_1$, for j = 0, 1, and 2.

(induction:) **Suppose that, for j = 0, 1, 2, …, m, $p_j = j\ p_1$.**

We will show that $p_{m+1} = (m+1)\ p_1$.

Therefore, $p_j = j\ p_1$ for all j.

That is, P(win the tournament) is prop. to your number of chips.

$p_m = 1/2\ p_{m-1} + 1/2\ p_{m+1}$. If $p_j = j\ p_1$ for $j \leq m$, then we have

$mp_1 = 1/2\ (m-1)p_1 + 1/2\ p_{m+1}$,

so $p_{m+1} = 2mp_1 - (m-1)\ p_1 = (m+1)p_1$.

7. Doubling up. Again, P(winning) = your proportion of chips.

Theorem 7.6.7, p152, describes another simplified scenario.

Suppose you either double each hand you play, or go to zero, each with probability 1/2.

Again, P(win a tournament) is prop. to your number of chips.

Again, $p_0 = 0$, and $p_1 = 1/2\ p_2 = 1/2\ p_2$, so again, $p_2 = 2\ p_1$.

We have shown that, for j = 0, 1, and 2, $p_j = j\ p_1$.

***(induction:)* Suppose that, for $j \leq m$, $p_j = j\ p_1$.**

We will show that $p_{2m} = (2m)\ p_1$.

Therefore, $p_j = j\ p_1$ for all $j = 2^k$. That is, P(win the tournament) is prop. to # of chips.

This time, $p_m = 1/2\ p_0 + 1/2\ p_{2m}$. If $p_j = j\ p_1$ for $j \leq m$, then we have

$mp_1 = 0 + 1/2\ p_{2m}$, so $p_{2m} = 2mp_1$. Done.

In Theorem 7.6.8, p152, you have k of the n chips in play. Each hand, you gain 1 with prob. p, or lose 1 with prob. q=1-p.

Suppose $0<p<1$ and $p\neq 0.5$. Let $r = q/p$. Then P(you win the tournament) = $\mathbf{(1-r^k)/(1-r^n)}$.

The proof is again by induction, and is similar to the proof we did of Theorem 7.6.6.

8. Examples.

(Chen and Ankenman, 2006). Suppose that a \$100 winner-take-all tournament has $1024 = 2^{10}$ players. So, you need to double up 10 times to win. Winner gets \$102,400.

Suppose you have probability $p = 0.54$ to double up, instead of 0.5.

What is your expected profit in the tournament? (Assume only doubling up.)

Answer. P(winning) = 0.54^{10}, so exp. return = 0.54^{10} (\$102,400) = \$215.89. So exp. <u>profit</u> = \$115.89.

What if each player starts with 10 chips, and you gain a chip with $p = 54\%$ and lose a chip with $p = 46\%$? What is your expected profit?

Answer. $r = q/p = .46/.54 = .852$. P(you win) = $(1-r^{10})/(1-r^{10240})$ = 79.9%.

So exp. profit = .799(\$102400) - \$100 ~ \$81700.

Random Walk example.

Suppose you start with 1 chip at time 0 and that your tournament is like a simple random walk, but if you hit 0 you are done. P(you have not hit zero by time 47)?

We know that starting at 0, $P(Y_1 \neq 0, Y_2 \neq 0, \ldots, Y_{2n} \neq 0) = P(Y_{2n} = 0)$.

So, for a random walk starting at (0,0),

by symmetry $P(Y_1 > 0, Y_2 > 0, \ldots, Y_{48} > 0) = ½ P(Y_1 \neq 0, Y_2 \neq 0, \ldots, Y_{2n} \neq 0)$

$= ½ P(Y_{48} = 0) = ½$ Choose(48,24)$(½)^{48}$.

Also $P(Y_1 > 0, Y_2 > 0, \ldots, Y_{48} > 0) = P(Y_1 = 1, Y_2 > 0, \ldots, Y_{48} > 0)$

= P(start at 0 and win your first hand, and then stay above 0 for at least 47 more hands)

= P(start at 0 and win your first hand) x P(from (1,1), stay above 0 for ≥ 47 more hands)

= 1/2 P(starting with 1 chip, stay above 0 for at least 47 more hands).

So, multiplying both sides by 2,

P(starting with 1 chip, stay above 0 for at least 47 hands) = Choose(48,24)$(½)^{48}$

= 11.46%.