Stat 100a: Introduction to Probability.

<u>Outline for the day:</u>

1. Luck and skill in poker.
2. Ivey and Booth, bluffing and expected value.
3. Expected value and heads up with AA or 55.
4. Facts about expected value.
5. Bernoulli random variables.
6. Moment generating functions.
7. Assign teams and discuss projects.
8. Binomial random variables.
9. Geometric random variables.

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1. Luck and skill in poker. pp 71-79.

Let equity = your expected portion of the pot after the hand, assuming no future betting.
= your expected number of chips after the hand - chips you had before the hand
= pot * p, where p = your probability of winning if nobody folds.
I define luck as the equity gained during the dealing of the cards.
Skill = equity gained during the betting rounds.

Example.
You have Qc Qd. I have 10s 9s. Board is 10d 8c 7c 4c. Pot is $5.
The river is 2d, you bet $3, and I call.
On the river, how much equity did you gain by luck and how much by skill?

Equity gained by luck on river = your equity when 2d is exposed – your equity on turn
= 100% ($5) - 35/44 ($5) = $1.02.
Why 35/44? I can win with a 10, 9, 6, or J that is not a club. There are 1 + 2 + 3 + 3 = 9 of these cards, so the remaining 35 cards give you the win.

Equity gained by skill on river = your equity after all the betting is over - your equity when the 2d is dealt
= increase in pot on river * P(you win) - your cost
= $6 * 100% - $3 = $3.

Luck and skill in poker, continued.

Example.
You have Qc Qd. I have 10s 9s. Board is 10d 8c 7c 4c. Pot is \$5.
The river is 2d, you bet \$3, and I call.
On the river, how much equity did you gain by luck and how much by skill?

Alternatively, let x = the number of chips you have after your \$3 bet on the river.
Before this bet, you had x + \$3 chips.

Equity gained by skill on river = your equity after all the betting is over - your equity when the 2d is dealt
= your expected number of chips after all the betting is over – your expected number of chips when the 2d is dealt
= (100%)(x + \$11) – (100%)(x + \$3 + \$5)
= \$3.

Bluffing. Ivey and Booth.

3. Heads up with AA.

Dan Harrington says that, "with a hand like AA, you really want to be one-on-one." True or false?

* Best possible pre-flop situation is to be all in with AA vs A8, where the 8 is the same suit as one of your aces, in which case you're about 94% to win. (the 8 could equivalently be a 6,7, or 9.) If you are all in for $100, then your expected holdings afterwards are $188.

a) In a more typical situation: you have AA against TT. You're 80% to win, so your expected value is $160.

b) Suppose that, after the hand vs TT, you get QQ and get up against someone with A9 who has more chips than you do. The chance of you winning this hand is 72%, and the chance of you winning both this hand and the hand above is 58%, so your expected holdings after both hands are $232:
you have 58% chance of having $400, and 42% chance to have $0.

c) Now suppose instead that you have AA and are all in against 3 callers with A8, KJ suited, and 44. Now you're 58.4% to quadruple up. So your expected holdings after the hand are $234, and the situation is just like (actually slightly better than) #1 and #2 combined: 58.4% chance to hold $400, and 41.6% chance for $0.

* So, being all-in with AA against 3 players is much better than being all-in with AA against one player: in fact, it's about like having two of these lucky one-on-one situations.

What about with a low pair?

a) You have $100 and 55 and are up against A9. You are 56% to win, so your expected value is $112.

b) You have $100 and 55 and are up against A9, KJ, and QJs. Seems pretty terrible, doesn't it? But you have a probability of 27.3% to quadruple, so your expected value is

0.273 x $400 = $109. About the same as #1!

[For these probabilities, see

http://www.cardplayer.com/poker_odds/texas_holdem]

4. Notes about expected value.

For any random variable X and any constants a and b,

$E(aX + b) = aE(X) + b.$

Also, $E(X+Y) = E(X) + E(Y)$,

unless $E(X) = \infty$ and $E(Y) = -\infty$, in which case $E(X)+E(Y)$ is undefined.

Thus $\sigma^2 = E[(X-\mu)^2]$

$= E[(X^2 - 2\mu X + \mu^2)]$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2.$

5. Bernoulli Random Variables, ch. 5.1.

If X = 1 with probability p, and X = 0 otherwise, then X = *Bernoulli (p)*.

Probability mass function (pmf):

$P(X = 1) = p$

$P(X = 0) = q$, where $p+q = 100\%$.

If X is Bernoulli (p), then μ = E(X) = p, and $\sigma = \sqrt{(pq)}$.

For example, suppose X = 1 if you have a pocket pair next hand; X = 0 if not.

p = 5.88%. So, q = 94.12%.

[Two ways to figure out p:

(a) Out of choose(52,2) combinations for your two cards, 13 * choose(4,2) are pairs.

13 * choose(4,2) / choose(52,2) = 5.88%.

(b) Imagine *ordering* your 2 cards. No matter what your 1st card is, there are 51 equally likely choices for your 2nd card, and 3 of them give you a pocket pair. 3/51 = 5.88%.]

$\mu = E(X) = .0588$. SD = $\sigma = \sqrt{(.0588 * 0.9412)} = 0.235$.

6. Moment generating functions, ch. 4.7

Suppose X is a random variable. $E(X)$, $E(X^2)$, $E(X^3)$, etc. are the *moments* of X.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Take derivatives with respect to t of $ø_X(t)$ and evaluate at t=0 to get moments of X.

1st derivative $(d/dt)\, e^{tX} = X\, e^{tX}$, $\quad (d/dt)^2\, e^{tX} = X^2\, e^{tX}$, etc.

$(d/dt)^k\, E(e^{tX}) = E[(d/dt)^k\, e^{tX}] = E[X^k\, e^{tX}]$, (see p.84)

so $ø'_X(0) = E[X^1\, e^{0X}] = E(X)$,

$ø''_X(0) = E[X^2\, e^{0X}] = E(X^2)$, etc.

The moment gen. function $ø_X(t)$ uniquely characterizes the distribution of X.

So to show that X is, say, Poisson, you just need to show that it has the moment generating function of a Poisson random variable.

Also, if X_i are random variables with cdfs F_i, and $ø_{X_i}(t) \to ø(t)$, where $ø_X(t)$ is the moment generating function of X which has cdf F, then $X_i \to X$ in distribution, i.e. $F_i(y) \to F(y)$ for all y where F(y) is continuous, see p85.

Moment generating functions, continued.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Suppose X is Bernoulli (0.4). What is $ø_X(t)$?

$E(e^{tX}) = (0.6)\ (e^{t(0)}) + (0.4)\ (e^{t(1)}) = 0.6 + 0.4\ e^t$.

Suppose X is Bernoulli (0.4) and Y is Bernoulli (0.7) and X and Y are independent.

What is the distribution of XY?

$ø_{XY}(t) = E(e^{tXY}) = P(XY=0)\ (e^{t(0)}) + P(XY=1)(e^{t(1)})$

$= P(X=0 \text{ or } Y=0)\ (1) + P(X=1 \text{ and } Y=1)e^t$

$= [1 – P(X=1)P(Y=1)] + P(X=1)P(Y=1)e^t$

$= [1 – 0.4 \text{ x } 0.7] + 0.4\text{x}0.7e^t$

$= 0.72 + 0.28e^t$, which is the moment generating function of a Bernoulli (0.28) random variable. Therefore XY is Bernoulli (0.28).

What about $Z = \min\{X,Y\}$?

Z = XY in this case, since X and Y are 0 or 1, so the answer is the same.

7. Assign teams of 3.

The project is problem 8.2, p165.

You need to write code to go all in or fold. In R, try:

```
install.packages(holdem)
library(holdem)
library(help="holdem")
```

timemachine, tommy, ursula, vera, william, and xena are examples.

crds1[1,1] is your higher card (2-14).

crds1[2,1] is your lower card (2-14).

crds1[1,2] and crds1[2,2] are suits of your higher card & lower card.

```
help(tommy)
tommy
```

```
function (numattable1, crds1, board1, round1, currentbet, mychips1,
  pot1, roundbets, blinds1, chips1, ind1, dealer1, tablesleft)
{ a1 = 0
  if (crds1[1, 1] == crds1[2, 1])
     a1 = mychips1
  a1
}
```

help(vera)

All in with a pair, any suited cards, or if the smaller card is at least 9.

```
function (numattable1, crds1, board1, round1, currentbet, mychips1,
   pot1, roundbets, blinds1, chips1, ind1, dealer1, tablesleft)
{a1 = 0
 if ((crds1[1, 1] == crds1[2, 1]) || (crds1[1, 2] == crds1[2,2]) ||
     (crds1[2, 1] > 8.5)) a1 = mychips1
   a1
}
```

You need to email me your function, to frederic@stat.ucla.edu. It should be written (or cut and pasted) simply into the body of the email. If you write it in Word, save as text first, and then paste it into the email. Just submit one email per team.

For instance, if your letter is “b”, you might do:

```
bruin = function (numattable1, crds1, board1, round1, currentbet,
    mychips1, pot1, roundbets, blinds1, chips1, ind1, dealer1,
    tablesleft) {
## all in with any pair higher than 7s, or if lower card is J or higher
a1 = 0
if ((crds1[1, 1] == crds1[2, 1]) && (crds1[1, 1] > 6.5)) a1 = mychips1
if (crds1[2,1] > 10.5) a1 = mychips1
a1
} ## end of bruin
```

8. Binomial Random Variables, ch. 5.2.

Suppose now X = # of times something with prob. p occurs, out of n independent trials

Then X = *Binomial (n.p)*.

e.g. the number of pocket pairs, out of 10 hands.

Now X could = 0, 1, 2, 3, …, or n.

pmf: $P(X = k) = \text{choose}(n, k) * p^k q^{n-k}$.

e.g. say n=10, k=3: $P(X = 3) = \text{choose}(10,3) * p^3 q^7$.

Why? Could have 1 1 1 0 0 0 0 0 0 0, or 1 0 1 1 0 0 0 0 0 0, etc.

choose(10, 3) choices of places to put the 1's, and for each the prob. is $p^3 q^7$.

Key idea: $X = Y_1 + Y_2 + \ldots + Y_n$, where the Y_i are independent and *Bernoulli* (p).

If X is Bernoulli (p), then $\mu = p$, and $\sigma = \sqrt{(pq)}$.

If X is Binomial (n,p), then $\mu = np$, and $\sigma = \sqrt{(npq)}$.

Binomial Random Variables, continued.

Suppose X = the number of pocket pairs you get in the next 100 hands.

<u>What's P(X = 4)? What's E(X)? σ?</u> X = Binomial (100, 5.88%).

$P(X = k) = \text{choose}(n, k) * p^k q^{n-k}$.

So, $P(X = 4) = \text{choose}(100, 4) * 0.0588^4 * 0.9412^{96} = 13.9\%$, or 1 in **7.2.**

$E(X) = np = 100 * 0.0588 =$ **5.88**. $\sigma = \sqrt{(100 * 0.0588 * 0.9412)} =$ **2.35**.

So, out of 100 hands, you'd *typically* get about 5.88 pocket pairs, +/- around 2.35.

9. Geometric Random Variables, ch 5.3.

Suppose now X = # of trials until the *first* occurrence.

(Again, each trial is independent, and each time the probability of an occurrence is p.)

Then X = *Geometric (p)*.

e.g. the number of hands til you get your next pocket pair.

[*Including the hand where you get the pocket pair. If you get it right away, then X = 1.*]

Now X could be 1, 2, 3, …, up to ∞.

pmf: $P(X = k) = p^1 q^{k-1}$.

e.g. say k=5: $P(X = 5) = p^1 q^4$. Why? Must be 0 0 0 0 1. Prob. = q * q * q * q * p.

If X is Geometric (p), then $\mu = 1/p$, and $\sigma = (\sqrt{q}) \div p$.

e.g. Suppose X = the number of hands til your next pocket pair. P(X = 12)? E(X)? σ?

X = Geometric (5.88%).

$P(X = 12) = p^1 q^{11}$ = 0.0588 * 0.9412 ^ 11 = **3.02%.**

E(X) = 1/p = **17.0**. σ = sqrt(0.9412) / 0.0588 = **16.5**.

So, you'd typically *expect* it to take 17 hands til your next pair, +/- around 16.5 hands.