Stat 100a: Introduction to Probability.

Outline for the day:

1. Review list.
2. Markov and Chebyshev examples.
3. More luck and skill examples, and other examples.
4. Binomial random variables.
5. Geometric random variables.
6. Negative binomial random variables.
7. Poisson random variables.
8. Continuous random variables and densities.
9. Exponential random variables.
10. Uniform, random variables.
11. Normal random variables.
12. Moment generating functions of different distributions.
13. Survivor functions.

Read through chapter 6.4! ♣ ♥ ◆ ♠

1. Review list for the midterm.

a. Meaning of probability and axioms of probability.
b. Basic multiplicative principle of counting.
c. Permutations
d. combinations.
e. Conditional probability.
f. Independence.
g. Multiplication rules.
h. Odds ratios.
i. cmf, cdf, and pdf.
j. Expected value, E(aX+b) and E(X+Y).
k. Bayes's rule.
l. Pot odds.
m. SD and variance.
n. Markov and Chevyshev inequalities.
o. Luck and skill.
p. Moment generating functions.

2. Markov and Chebyshev examples.

Suppose X is the time til someone goes all in and gets called, and suppose $E(X) = 30$ minutes and $V(X) = 20$ min. What does the Markov inequality tell you about $P(X \geq 50 \text{ minutes})$?

X is non-negative, so we can use the Markov inequality.
$P(X \geq c) \leq E(X)/c$. Here $E(X) = 30$ min, and $c = 50$ min, and $30/50 = 0.6$,
so the Markov inequality tells us $P(X \geq 50) \leq 60\%$.

What does the Chebyshev inequality tell you about $P(X > 40 \text{ or } X < 20)$?

For any random variable Y with expected value μ and variance σ^2, and any real number $a > 0$,

$$P(|Y - \mu| \geq a) \leq \sigma^2 / a^2.$$

Here $\mu = 30$, $a = 10$. $P(X > 40 \text{ or } X < 20) = P(|Y - \mu| \geq 10)$
$\leq 20/10^2 = 20\%$.
So $P(X > 40 \text{ or } X < 20) \leq 20\%$.

3. More luck and skill examples, and other probability examples.

Players A and B are heads up. A has A♣ 3♣. B has 5♥ 4♥. The pot is 100.
The flop is 5♠ 5◆ 4♠.
A checks, B bets 100, and A calls.
The turn is K◆.

a. When the flop is revealed, what is A's chance of winning?
b. How much equity did player A gain due to luck on the turn?
c. How much expected profit did player A gain due to skill on the flop?

a. Must be AA. P(A winning) = C(3,2)/C(45,2) = 3/990.

b. A's equity gain due to K◆ on the turn = A's equity after K◆ – A's equity before K◆.
= (0%)(300) – 3/990(300) = -0.91.

c. A's profit gain during flop betting = (A's equity after betting – cost to A) – A's equity before betting
= [(300)(3/990) – 100] – (100)(3/990) = -99.39.

♣ ♥ ◆ ♠

More luck and skill examples, continued.
https://www.youtube.com/watch?v=1l5d8b6zF8c
Example 2.1.5. Duhamel versus Affleck. In the 2010 WSOP Main Event, with 15 of the original 7319 players left, chip leader Jonathan Duhamel raised from 200,000 to 575,000 with J♣ J♥. Matt Affleck reraised to 1,550,000 with A♣ A♠. Duhamel reraised to 3,925,000. Affleck called. The flop came 10♦ 9♣ 7♥, Duhamel checked, Affleck bet 5,000,000, and Duhamel called. The turn was the Q♦, Duhamel checked, Affleck went all in for 11,630,000, and Duhamel faced a tough decision. Given the board and the players' cards, if Duhamel calls, what is his probability of winning the hand?

In order to win, Duhamel needs the river to be a king, jack, or 8. The 44 possible river cards are each equally likely, and there are 4 kings + 2 jacks + 4 8s = 10 cards remaining, so Duhamel's chance of winning is 10/44 ~ 22.73%.

How much expected profit did Affleck gain due to skill on the turn?
Before the bet on the turn, the pot was 3,925,000 x 2 + 10,000,000 = 17,850,000.
Affleck's equity was 34/44 x 17,850,000 ~ 13.8 million.
The betting on the turn increased Affleck's equity to 34/44 x 41,110,000 ~ 31.8 million, which is an increase of 18 million, but Affleck has put in 11,630,000 on the turn, so his expected profit is 18 million – 11.63 million = 6.37 million.

How much expected profit, or equity, did Affleck gain due to luck on the river? -31.8 million.

A face card is a J, Q, or K. In one hand of texas holdem, what is the prob. you are dealt pocket kings, given that both your cards are face cards?

Using the def. of conditional probability,
P(KK | both face cards) = P(KK & both face cards) / P(both face cards).
Note that P(KK and both face cards) = P(KK), because if they are both kings then they are both face cards.
So P(KK | both face cards) = P(KK) / P(both face cards)
$= C(4,2)/C(52,2) \div C(12,2)/C(52,2)$
$= 1/11 \sim 9.09\%$.

What is P(you are dealt at least one king)?
At least one → 1 minus trick.
P(at least one K) = 1 – P(no kings)
$= 1 - C(48,2)/C(52,2)$
$\sim 14.9\%$.

4. Binomial Random Variables, ch. 5.2.

Suppose now X = # of times something with prob. p occurs, out of n independent trials

Then X = *Binomial (n.p)*.

e.g. the number of pocket pairs, out of 10 hands.

Now X could = 0, 1, 2, 3, …, or n.

pmf: $P(X = k) = \text{choose}(n, k) * p^k q^{n-k}$.

e.g. say n=10, k=3: $P(X = 3) = \text{choose}(10,3) * p^3 q^7$.

Why? Could have 1 1 1 0 0 0 0 0 0 0, or 1 0 1 1 0 0 0 0 0 0, etc.

choose(10, 3) choices of places to put the 1's, and for each the prob. is $p^3 q^7$.

Key idea: $X = Y_1 + Y_2 + \ldots + Y_n$, where the Y_i are independent and *Bernoulli* (p).

If X is Bernoulli (p), then $\mu = p$, and $\sigma = \sqrt{(pq)}$.

If X is Binomial (n,p), then $\mu = np$, and $\sigma = \sqrt{(npq)}$.

Binomial Random Variables, continued.

Suppose X = the number of pocket pairs you get in the next 100 hands.

<u>What's P(X = 4)? What's E(X)? σ?</u>   X = Binomial (100, 5.88%).

$P(X = k) = \text{choose}(n, k) * p^k q^{n - k}$.

So, $P(X = 4) = \text{choose}(100, 4) * 0.0588^4 * 0.9412^{96} = 13.9\%$, or 1 in **7.2.**

$E(X) = np = 100 * 0.0588 =$ **5.88**. $\sigma = \sqrt{(100 * 0.0588 * 0.9412)} =$ **2.35**.

So, out of 100 hands, you'd *typically* get about 5.88 pocket pairs, +/- around 2.35.

5. Geometric Random Variables, ch 5.3.

Suppose now X = # of trials until the *first* occurrence.

(Again, each trial is independent, and each time the probability of an occurrence is p.)

Then X = *Geometric (p)*.

e.g. the number of hands til you get your next pocket pair.

[*Including the hand where you get the pocket pair. If you get it right away, then X = 1.*]

Now X could be 1, 2, 3, …, up to ∞.

pmf: $P(X = k) = p^1 q^{k-1}$.

e.g. say k=5: $P(X = 5) = p^1 q^4$. Why? Must be 0 0 0 0 1. Prob. = q * q * q * q * p.

If X is Geometric (p), then $\mu = 1/p$, and $\sigma = (\sqrt{q}) \div p$.

e.g. Suppose X = the number of hands til your next pocket pair. P(X = 12)? E(X)? σ?

X = Geometric (5.88%).

$P(X = 12) = p^1 q^{11}$ = 0.0588 * 0.9412 ^ 11 = **3.02%.**

E(X) = 1/p = **17.0**. σ = sqrt(0.9412) / 0.0588 = **16.5**.

So, you'd typically *expect* it to take 17 hands til your next pair, +/- around 16.5 hands.

6. Negative Binomial Random Variables, ch 5.4.

Recall: if each trial is independent, and each time the probability of an occurrence is p, and X = # of trials until the *first* occurrence, then:

X is Geometric (p), $P(X = k) = p^1 q^{k-1}$, $\mu = 1/p$, $\sigma = (\sqrt{q}) \div p$.

Suppose now X = # of trials until the *rth* occurrence.

Then X = *negative binomial (r,p)*.

e.g. the number of hands you have to play til you've gotten r=3 pocket pairs.

Now X could be 3, 4, 5, …, up to ∞.

pmf: $P(X = k) = \text{choose}(k-1, r-1)\ p^r q^{k-r}$, for k = r, r+1, ….

e.g. say r=3 & k=7: $P(X = 7) = \text{choose}(6,2)\ p^3 q^4$.

Why? Out of the first 6 hands, there must be exactly r-1 = 2 pairs. Then pair on 7th.

P(exactly 2 pairs on first 6 hands) = $\text{choose}(6,2)\ p^2 q^4$. P(pair on 7th) = p.

If X is negative binomial (r,p), then $\mu = r/p$, and $\sigma = (\sqrt{rq}) \div p$.

e.g. Suppose X = the number of hands til your 12th pocket pair. P(X = 100)? E(X)? σ?

X = Neg. binomial (12, 5.88%).

$P(X = 100) = \text{choose}(99,11)\ p^{12} q^{88}$

= choose(99,11) * 0.0588 ^ 12 * 0.9412 ^ 88 = **0.104%.**

E(X) = r/p = 12/0.0588 ~ **204**. σ = sqrt(12*0.9412) / 0.0588 = **57.2**.

So, you'd typically *expect* it to take 204 hands til your 12th pair, +/- around 57.2 hands.

7. Poisson random variables, ch 5.5.

Suppose Player 1 plays in a slow game, where about 15 hands are dealt per hour, and she decides to do a big bluff whenever the second hand on her watch, at the start of the deal, is in some predetermined 4 second interval.

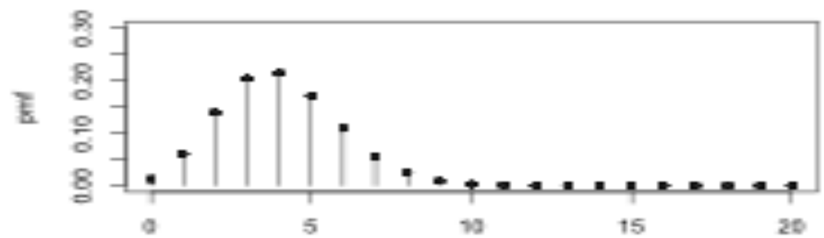
Player 2 plays in a faster game where about 20 hands are dealt per hour, and she bluffs only when the second hand on her watch at the start of the deal is in a 3 second interval.

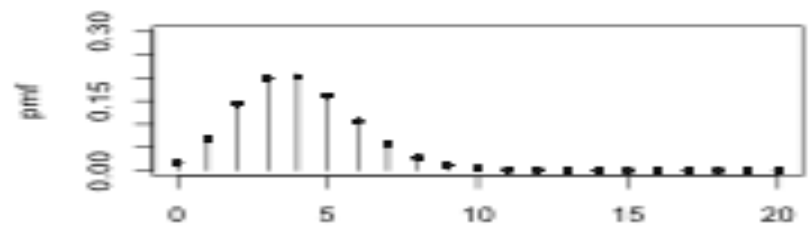
Player 3 plays in a fast game with 30 hands per hour, and bluffs only when the second hand is in a 2 sec interval.

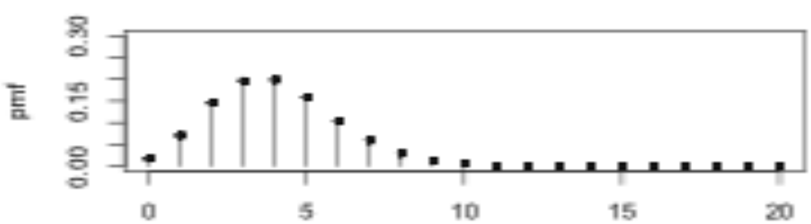
Each of the three players will thus average one bluff every hour.

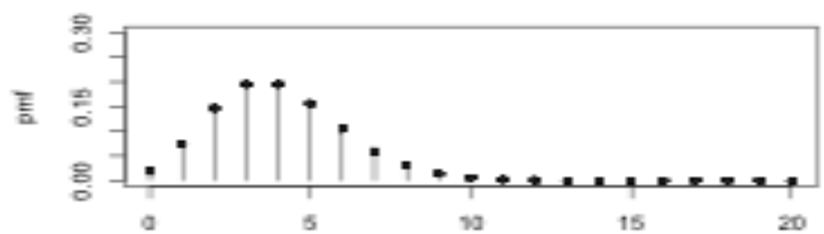
Let X_1, X_2, and X_3 denote the number of big bluffs attempted in a given 4 hour interval by Player 1, Player 2, and Player 3, respectively.
Each of these random variables is binomial with an expected value of 4, and a variance approaching 4.
They are converging toward some limiting distribution, and that limiting distribution is called the *Poisson* distribution.

They are converging toward some limiting distribution, and that limiting distribution is called the *Poisson* distribution. Unlike the binomial distribution which depends on two parameters, n and p, the Poisson distribution depends only on one parameter, λ, which is called the *rate*. In this example, $\lambda = 4$.









The pmf of the Poisson random variable is $f(k) = e^{-\lambda}\lambda^k/k!$, for $k=0,1,2,\ldots$, and for $\lambda > 0$, with the convention that $0!=1$, and where $e = 2.71828\ldots$.
The Poisson random variable is the limit in distribution of the binomial distribution as $n \to \infty$ while np is held constant.

For a Poisson(λ) random variable X, $E(X) = \lambda$, and $Var(X) = \lambda$ also. $\lambda = rate$.

Example. Suppose in a certain casino jackpot hands are defined so that they tend to occur about once every 50,000 hands on average. If the casino deals approximately 10,000 hands per day, **a)** what are the expected value and standard deviation of the number of jackpot hands dealt in a 7 day period? **b)** How close are the answers using the binomial distribution and the Poisson approximation? Using the Poisson model, if X represents the number of jackpot hands dealt over this week, what are **c)** $P(X = 5)$ and **d)** $P(X = 5 \mid X > 1)$?

Answer. It is reasonable to assume that the outcomes on different hands are iid, and this applies to jackpot hands as well. In a 7 day period, approximately 70,000 hands are dealt, so X = the number of occurrences of jackpot hands is binomial$(n=70{,}000, p=1/50{,}000)$. Thus **a)** $E(X) = np = 1.4$, and $SD(X) = \sqrt{(npq)} = \sqrt{(70{,}000 \times 1/50{,}000 \times 49{,}999/50{,}000)} \sim 1.183204$. **b)** Using the Poisson approximation, $E(X) = \lambda = np = 1.4$, and $SD(X) = \sqrt{\lambda} \sim 1.183216$. The Poisson model is a very close approximation in this case. Using the Poisson model with rate $\lambda = 1.4$,

c) $P(X=5) = e^{-1.4}\, 1.4^5/5! \sim 1.105\%$.

d) $P(X = 5 \mid X > 1) = P(X = 5 \text{ and } X > 1) \div P(X > 1) = P(X = 5) \div P(X>1) = [e^{-1.4}\ 1.4^5/5!] \div [1 - e^{-1.4}\ 1.4^0/0! - e^{-1.4}\, 1.4^1/1!] \sim 2.71\%$.

8. Continuous random variables and their densities, p103-107.

Density (or pdf = Probability Density Function) f(y):

$$\int_B f(y)\, dy = P(X \text{ in } B).$$

Expected value, $\mu = E(X) = \int y\, f(y)\, dy$. (= $\sum y\, P(y)$ for discrete X.)

Variance, $\sigma^2 = V(X) = E(X^2) - \mu^2$.

$SD(X) = \sqrt{V(X)}$.

For examples of pdfs, see p104, 106, and 107.

9. Exponential distribution, ch 6.4.

Useful for modeling waiting times til something happens (like the geometric).

pdf of an exponential random variable is $f(y) = \lambda\ exp(-\lambda\ y)$, for $y \geq 0$, and $f(y) = 0$ otherwise.
The cdf is $F(y) = 1 - exp(-\lambda\ y)$, for $y \geq 0$.
If X is exponential with parameter λ, then $E(X) = SD(X) = 1/\lambda$

If the total numbers of events in any disjoint time spans are independent, then these totals are Poisson random variables. If in addition the events are occurring at a constant rate λ, then the times between events, or *interevent times*, are exponential random variables with mean $1/\lambda$.

Example. Suppose you play 20 hands an hour, with each hand lasting exactly 3 minutes, and let X be the time in hours until the end of the first hand in which you are dealt pocket aces. Use the exponential distribution to approximate $P(X \leq 2)$ and compare with the exact solution using the geometric distribution.

Answer. Each hand takes 1/20 hours, and the probability of being dealt pocket aces on a particular hand is 1/221, so the rate λ = 1 in 221 hands = 1/(221/20) hours ~ 0.0905 per hour.
Using the exponential model, $P(X \le 2 \text{ hours}) = 1 - \exp(-2\lambda) \sim 16.556\%$.
This is an approximation, however, since by assumption X is not continuous but must be an integer multiple of 3 minutes.
Let Y = the number of hands you play until you are dealt pocket aces. Using the geometric distribution, $P(X \le 2 \text{ hours}) = P(Y \le 40 \text{ hands})$
$= 1 - (220/221)^{40} \sim 16.590\%$.

The survivor function for an exponential random variable is particularly simple: $P(X > c) = \int_c^\infty f(y)dy = \int_c^\infty \lambda \exp(-\lambda y)dy = -\exp(-\lambda y)]_c^\infty = \exp(-\lambda c)$.

Like geometric random variables, exponential random variables have the *memorylessness* property: if X is exponential, then for any non-negative values a and b, $P(X > a+b \mid X > a) = P(X > b)$. *(See p115)*.
Thus, with an exponential (or geometric) random variable, if after a certain time you still have not observed the event you are waiting for, then the distribution of the *future,* additional waiting time until you observe the event is the same as the distribution of the *unconditional* time to observe the event to begin with.

10. Uniform Random Variables and R.

Continuous random variables are often characterized by their

probability density functions (pdf, or *density*): a function f(x)

such that P{X is in B} = $\int_B f(x)\,dx$.

Uniform: f(x) = c, for x in (a, b).

= 0, for all other x.

[Note: c must = 1/(b-a), so that $\int_a^b f(x)\,dx$ = P{X is in (a,b)} = 1.]
Uniform (0,1). See p107-109.
f(y) = 1, for y in (0,1). $\mu = 0.5$. $\sigma \sim 0.29$.
P(X is between 0.4 and 0.6) = $\int_{.4}^{.6} f(y)\,dy = \int_{.4}^{.6} 1\,dy = 0.2$.

In R, runif(1,min=a,max=b) produces a pseudo-random uniform.

11. Normal random variables.

So far we have seen two continuous random variables, the uniform and the exponential.

Normal. pp 115-117. mean = μ, SD = σ, $f(y) = 1/\sqrt{(2\pi\sigma^2)}\ e^{-(y-\mu)^2/2\sigma^2}$.

Symmetric around μ,

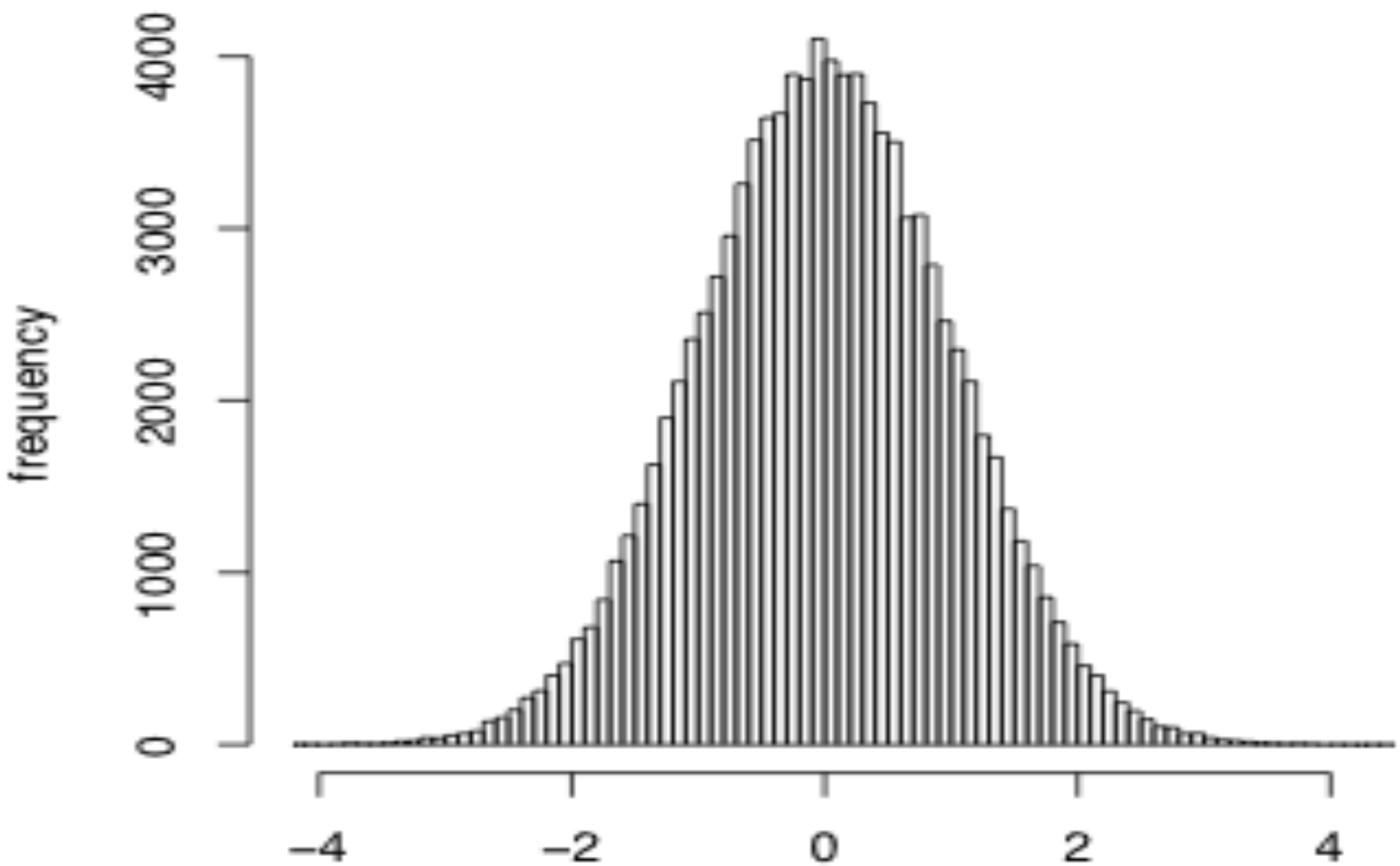
50% of the values are within 0.674 SDs of μ,

68.27% of the values are within 1 SD of μ, and

95% are within 1.96 SDs of μ.

* Standard Normal. Normal with $\mu = 0$, $\sigma = 1$. See pp 117-118.

Standard normal density:
68.27% between -1.0 and 1.0
95% between -1.96 and 1.96



12. Moment generating functions of some random variables.

Bernoulli(p). $\phi_X(t) = pe^t + q$.

Binomial(n,p). $\phi_X(t) = (pe^t + q)^n$. p94.

Geometric(p). $\phi_X(t) = pe^t/(1 - qe^t)$.

Neg. binomial (r,p). $\phi_X(t) = [pe^t/(1 - qe^t)]^r$. p97.

Poisson(λ). $\phi_X(t) = e^{\{\lambda e^t - \lambda\}}$. p100.

Uniform (a,b). $\phi_X(t) = (e^{tb} - e^{ta})/[t(b-a)]$. p108.

Exponential (λ). $\phi_X(t) = \lambda/(\lambda - t)$. p123.

Normal. $\phi_X(t) = e^{\{t\mu + t^2\sigma^2/2\}}$.

13. Survivor functions. p96 and 115.

Recall the cdf $F(b) = P(X \leq b)$.
The survivor function is $S(b) = P(X > b) = 1 - F(b)$.
Some random variables have really simple survivor functions and it can be convenient to work with them.
If X is geometric, then $S(b) = P(X > b) = q^b$, for $b = 0,1,2,\ldots$.
For instance, let b=2. $X > 2$ means the 1st two were misses,
i.e. $P(X>2) = q^2$.
For exponential X, $F(b) = 1 - \mathit{exp}(-\lambda b)$, so $S(b) = \mathit{exp}(-\lambda b)$.

An interesting fact is that, if X takes on only values in $\{0,1,2,3,\ldots\}$,
then $E(X) = S(0) + S(1) + S(2) + \ldots$.
Proof. See p96.

$$
\begin{aligned}
S(0) &= P(X=1) + P(X=2) + P(X=3) + P(X=4) + \ldots \\
S(1) &= \phantom{P(X=1) + {}} P(X=2) + P(X=3) + P(X=4) + \ldots \\
S(2) &= \phantom{P(X=1) + P(X=2) + {}} P(X=3) + P(X=4) + \ldots \\
S(3) &= \phantom{P(X=1) + P(X=2) + P(X=3) + {}} P(X=4) + \ldots
\end{aligned}
$$

Add these up and you get
$0\,P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) + 4P(X=4) + \ldots$
$= \sum kP(X=k) = E(X)$.