Stat 100a: Introduction to Probability.

Outline for the day:

1. Hand in HW3.
2. Return midterm 2.
3. Avoiding zero.
4. Chip proportions and induction.
5. Doubling up.
6. Doubling up example.
7. RW example.
8. Another RW example.

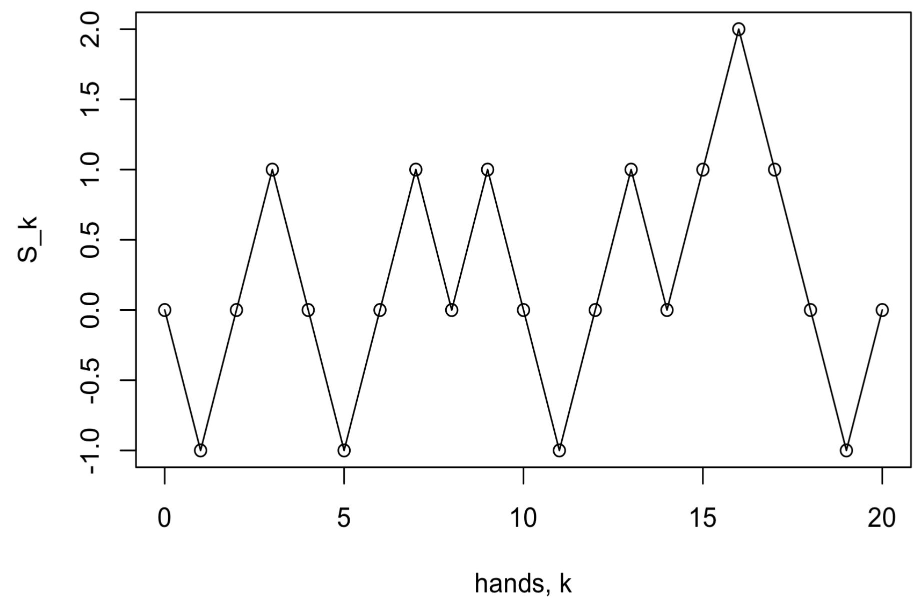
Computer project is due Jul 27 8pm.
Read through ch7.

1. Hand in homework 3!

2. Please be silent while I return midterm 2

in alphabetical order by last name.

Mean = 90, SD = 15. Median = 90.



3. Avoiding zero.

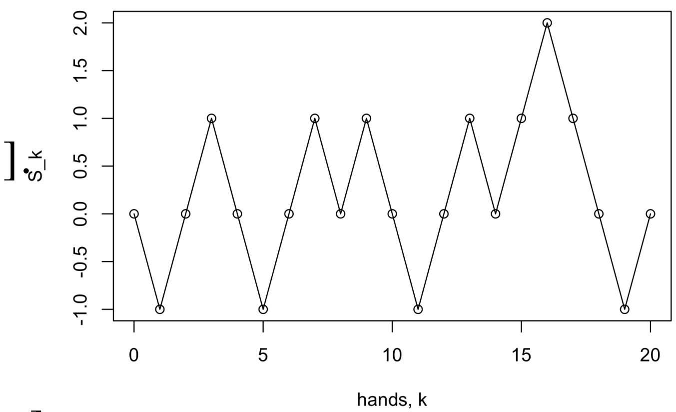
For a simple random walk, for any even # n, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.

Proof. The number of paths from (0,0) to (n, j) that don't touch the x-axis at positive times

= the number of paths from (1,1) to (n,j) that don't touch the x-axis at positive times

= paths from (1,1) to (n,j) - paths from (1,-1) to (n,j) by the *reflection principle*

$= N_{n-1,j-1} - N_{n-1,j+1}$



Let $Q_{n,j} = P(S_n = j)$.

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = j) = ½[Q_{n-1,j-1} - Q_{n-1,j+1}]$.

Summing from j = 2 to ∞,

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n > 0)$

$= ½[Q_{n-1,1} - Q_{n-1,3}] + ½[Q_{n-1,3} - Q_{n-1,5}] + ½[Q_{n-1,5} - Q_{n-1,7}] + \ldots$

$= (1/2)\, Q_{n-1,1}$

$= (1/2)\, P(S_n = 0)$, because to end up at (n, 0), you have to be at (n-1,+/-1),

so $P(S_n = 0) = (1/2)\, Q_{n-1,1} + (1/2)\, Q_{n-1,-1} = Q_{n-1,1}$.

By the same argument, $P(S_1 < 0, S_2 < 0, \ldots, S_{n-1} < 0, S_n < 0) = (1/2)\, P(S_n = 0)$.

So, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.

4. Chip proportions and induction, Theorem 7.6.6.

P(win a tournament) is proportional to your number of chips.

Simplified scenario. Suppose you either go up or down 1 each hand, with prob. 1/2.

Suppose there are n chips, and you have k of them.

Let p_k = P(win tournament given k chips) = P(random walk goes k -> n before hitting 0).

Now, clearly $p_0 = 0$. Consider p_1. From 1, you will either go to 0 or 2.

So, $p_1 = 1/2\ p_0 + 1/2\ p_2 = 1/2\ p_2$. That is, $p_2 = 2\ p_1$.

We have shown that $p_j = j\ p_1$, for j = 0, 1, and 2.

***(induction:)* Suppose that, for j = 0, 1, 2, …, m, $p_j = j\ p_1$.**

We will show that $p_{m+1} = (m+1)\ p_1$.

Therefore, $p_j = j\ p_1$ for all j.

That is, P(win the tournament) is prop. to your number of chips.

$p_m = 1/2\ p_{m-1} + 1/2\ p_{m+1}$. If $p_j = j\ p_1$ for $j \le m$, then we have

$mp_1 = 1/2\ (m-1)p_1 + 1/2\ p_{m+1}$,

so $p_{m+1} = 2mp_1 - (m-1)\ p_1 = (m+1)p_1$.

5. Doubling up. Again, P(winning) = your proportion of chips.
Theorem 7.6.7, p152, describes another simplified scenario.
Suppose you either double each hand you play, or go to zero, each with probability 1/2.

Again, P(win a tournament) is prop. to your number of chips.

Again, $p_0 = 0$, and $p_1 = 1/2\ p_2 = 1/2\ p_2$, so again, $p_2 = 2\ p_1$.

We have shown that, for j = 0, 1, and 2, $p_j = j\ p_1$.

***(induction:)* Suppose that, for $j \le m$, $p_j = j\ p_1$.**

We will show that $p_{2m} = (2m)\ p_1$.

Therefore, $p_j = j\ p_1$ for all $j = 2^k$. That is, P(win the tournament) is prop. to # of chips.

This time, $p_m = 1/2\ p_0 + 1/2\ p_{2m}$. If $p_j = j\ p_1$ for $j \le m$, then we have

$mp_1 = 0 + 1/2\ p_{2m}$, so $p_{2m} = 2mp_1$. Done.

Problem 7.14 refers to Theorem 7.6.8, p152.

You have k of the n chips in play. Each hand, you gain 1 with prob. p, or lose 1 with prob. q=1-p.

Suppose $0<p<1$ and $p \neq 0.5$. Let $r = q/p$. Then P(you win the tournament) = $\mathbf{(1-r^k)/(1-r^n)}$.

The proof is again by induction, and is similar to the proof we did of Theorem 7.6.6.

6. Doubling up example. (Chen and Ankenman, 2006). Suppose that a $100 winner-take-all tournament has $1024 = 2^{10}$ players. So, you need to double up 10 times to win. Winner gets $102,400.

Suppose you have probability $p = 0.54$ to double up, instead of 0.5.

What is your expected profit in the tournament? (Assume only doubling up.)

P(winning) $= 0.54^{10}$, so exp. return $= 0.54^{10}$ ($102,400) = $215.89. So exp. <u>profit</u> = $115.89.

7. Random Walk example.

Suppose you start with 1 chip at time 0 and that your tournament is like a simple random walk, but if you hit 0 you are done. P(you have not hit zero by time 47)?

We know that starting at 0, $P(Y_1 \neq 0, Y_2 \neq 0, \ldots, Y_{2n} \neq 0) = P(Y_{2n} = 0)$.

So, $P(Y_1 > 0, Y_2 > 0, \ldots, Y_{48} > 0) = ½ P(Y_{48} = 0) = ½ \text{Choose}(48,24)(½)^{48}$

$= P(Y_1 = 1, Y_2 > 0, \ldots, Y_{48} > 0)$

= P(start at 0 and win your first hand, and then stay above 0 for at least 47 more hands)

= P(start at 0 and win your first hand) x P(from (1,1), stay above 0 for ≥ 47 more hands)

= 1/2 P(starting with 1 chip, stay above 0 for at least 47 more hands).

So, P(starting with 1 chip, stay above 0 for at least 47 hands) = $\text{Choose}(48,24)(½)^{48}$

= 11.46%.