Stat 100a: Introduction to Probability.

<u>Outline for the day:</u>

1. More about 3 of a kind.
2. Heads up with AA or a small pair.
3. Bayes's rule.
4. Pot odds.
5. Variance and SD.
6. Bernoulli random variables.
7. Moment generating functions.
8. Independent random variables.
9. Binomial random variables.

Read through ch5 for next week!

We've already covered the material for hw1 due Jan 29.

UCLA Statistics Club offers free tutoring for lower division statistics courses.
Students from other courses are welcome to attend but priority is for STATS 10, 13, and 20.
Boelter 4413, Mondays and Wednesdays, 4-5 PM.

1. More about P(flop 3 of a kind) = 13 * C(4,3) * C(12,2) * 4 * 4 / C(52,5).

choose(52,2) is how you do C(52,2) in R and 4 * 4 is how you do 4 x 4.

Why C(12,2) and not 12 * 11?
If we said 12 * 11, then we would be doublecounting each outcome like 33347. Once we have picked the triplet, there are 12 choices for a number to go with it, and then once we have picked such a number, there would be 11 choices to go with it too. But we are counting combinations here, so we are ignoring the order of the 5 cards, and thus 33347 is equivalent to 33374 and we don't want to count both separately. Thus, for each triplet such as 333, we need two distinct numbers to go with it and there are choose(12,2) such choices of two other distinct numbers.

2. Heads up with AA.

Dan Harrington says that, "with a hand like AA, you really want to be one-on-one." True or false?

* Best possible pre-flop situation is to be all in with AA vs A8, where the 8 is the same suit as one of your aces, in which case you're about 94% to win. (the 8 could equivalently be a 6,7, or 9.) If you are all in for $100, then your expected holdings afterwards are $188.

a) In a more typical situation: you have AA against TT. You're 80% to win, so your expected value is $160.

b) Suppose that, after the hand vs TT, you get QQ and get up against someone with A9 who has more chips than you do. The chance of you winning this hand is 72%, and the chance of you winning both this hand and the hand above is 58%, so your expected holdings after both hands are $232:
you have 58% chance of having $400, and 42% chance to have $0.

c) Now suppose instead that you have AA and are all in against 3 callers with A8, KJ suited, and 44. Now you're 58.4% to quadruple up. So your expected holdings after the hand are $234, and the situation is just like (actually slightly better than) #1 and #2 combined: 58.4% chance to hold $400, and 41.6% chance for $0.

* So, being all-in with AA against 3 players is much better than being all-in with AA against one player: in fact, it's about like having two of these lucky one-on-one situations.

What about with a low pair?

a) You have $100 and 55 and are up against A9. You are 56% to win, so your expected value is $112.

b) You have $100 and 55 and are up against A9, KJ, and QJs. Seems pretty terrible, doesn't it? But you have a probability of 27.3% to quadruple, so your expected value is

0.273 x $400 = $109. About the same as #1!

[For these probabilities, see http://www.cardplayer.com/poker_odds/texas_holdem]

3. Bayes's rule, p49-52.

Suppose that B_1, B_2, B_n are disjoint events and that exactly one of them must occur.

Suppose you want $P(B_1 | A)$, but you only know $P(A | B_1)$, $P(A | B_2)$, etc.,

and you also know $P(B_1), P(B_2), \ldots, P(B_n)$.

Bayes' Rule: If $B_{1,\ldots,} B_n$ are disjoint events with $P(B_1 \text{ or } \ldots \text{ or } B_n) = 1$, then

$P(B_i | A) = P(A | B_i) * P(B_i) \div [\sum P(A | B_j)P(B_j)]$.

Why? Recall: $P(X | Y) = P(X \& Y) \div P(Y)$. So $P(X \& Y) = P(X | Y) * P(Y)$.

$$P(B_1 | A) = P(A \& B_1) \div P(A)$$

$$= P(A \& B_1) \div [P(A \& B_1) + P(A \& B_2) + \ldots + P(A \& B_n)]$$

$$= P(A | B_1) * P(B_1) \div [P(A | B_1)P(B_1) + P(A | B_2)P(B_2) + \ldots + P(A | B_n)P(B_n)].$$

Bayes's rule, continued.

Bayes's rule: If $B_1, \ldots, B_n$ are disjoint events with $P(B_1 \text{ or } \ldots \text{ or } B_n) = 1$, then

$P(B_i \mid A) = P(A \mid B_i) * P(B_i) \div [\sum P(A \mid B_j)P(B_j)]$.

See example 3.4.1, p50. If a test is 95% accurate and 1% of the pop. has a condition, then given a random person from the population,

P(she has the condition | she tests positive)

= P(cond | +)

= P(+ | cond) P(cond) ÷ [P(+ | cond) P(cond) + P(+ | no cond) P(no cond)]

= 95% x 1% ÷ [95% x 1% + 5% x 99%]

~ 16.1%.

Tests for rare conditions must be extremely accurate.

Bayes' rule example.

Suppose P(your opponent has the nuts) = 1%, and P(opponent has a weak hand) = 10%. Your opponent makes a huge bet. Suppose she'd only do that with the nuts or a weak hand, and that P(huge bet | nuts) = 100%, and P(huge bet | weak hand) = 30%.

What is P(nuts | huge bet)?

P(nuts | huge bet) =

$$\frac{\text{P(huge bet | nuts) * P(nuts)}}{\text{P(huge bet | nuts) P(nuts) + P(huge bet | horrible hand) P(horrible hand)}}$$

$$= \frac{100\% * 1\%}{100\% * 1\% \quad + \quad 30\% * 10\%}$$

= **25%.**

4. POT ODDS CALCULATIONS.

Suppose someone bets (or raises) you, going all-in. What should your chances of winning be in order for you to correctly call?

Let B = the amount bet to you, i.e. the additional amount you'd need to put in if you want to call. So, if you bet 100 & your opponent with 800 left went all-in, B = 700.

Let POT = the amount in the pot right now (including your opponent's bet).

Let p = your probability of winning the hand if you call. So prob. of losing = 1-p.

Let CHIPS = the number of chips you have right now.

If you call, then E[your chips at end] = (CHIPS - B)(1-p) + (CHIPS + POT)(p)

= CHIPS(1-p+p) - B(1-p) + POT(p) = CHIPS - B + Bp + POTp

If you fold, then E[your chips at end] = CHIPS.

You want your expected number of chips to be maximized, so it's worth calling if

-B + Bp + POTp > 0, i.e. if **p > B / (B+POT).**

4. Pot odds and expected value, continued.

From previous slide, to call an all-in, need P(win) > B ÷ (B+pot).
Expressed as an *odds ratio*, this is sometimes referred to as *pot odds* or *express odds*.

If the bet is not all-in & another betting round is still to come, need
P(win) > wager ÷ (wager + winnings),
where winnings = pot + amount you'll win on later betting rounds,
wager = total amount you will wager including the current round & later rounds, assuming no folding.

The terms *Implied-odds* / *Reverse-implied-odds* describe the cases where *winnings > pot* or where *wager > B*, respectively. See p66.

Example: 2006 World Series of Poker (WSOP). ♣ ♥ ♦ ♠

Blinds: 200,000/400,000, + 50,000 ante.

Jamie Gold (4♣ 3♣): 60 million chips. Calls.

Paul Wasicka (8♠ 7♠): 18 million chips. Calls.

Michael Binger (A♥ 10♥): 11 million chips. Raises to $1,500,000.

Gold & Wasicka call. (pot = 4,650,000) Flop: 6♠ 10♣ 5♠.

- Wasicka checks, Binger bets $3,500,000. (pot = 8,150,000)
- Gold moves all-in for 16,450,000. (pot = 24,600,000)
- Wasicka folds. Q: Based on expected value, should he have called?

If Binger will fold, then Wasicka's chances to beat Gold must be at least 16,450,000 / (24,600,000 + 16,450,000) = 40.1%.

If Binger calls, it's a bit complicated, but basically Wasicka's chances must be at least 16,450,000 / (24,600,000 + 16,450,000 + 5,950,000) = 35.0%.

5. Variance and SD.

Expected Value: $E(X) = \mu = \sum k\ P(X=k)$.

Variance: $V(X) = \sigma^2 = E[(X-\mu)^2]$. Turns out this $= E(X^2) - \mu^2$.

Standard deviation $= \sigma = \sqrt{V(X)}$. Indicates how far an observation would *typically* deviate from μ.

Examples:

<u>Game 1.</u> Say X = \$4 if red card, X = \$-5 if black.

E(X) = (\$4)(0.5) + (\$-5)(0.5) = -\$0.50.

$E(X^2)$ = (\$$4^2$)(0.5) + (\$-5^2)(0.5) = (\$16)(0.5) + (\$25)(0.5) = \$20.5.

So $\sigma^2 = E(X^2) - \mu^2$ = \$20.5 - \$-0.50^2 = \$20.25. **σ = \$4.50**.

<u>Game 2.</u> Say X = \$1 if red card, X = \$-2 if black.

E(X) = (\$1)(0.5) + (\$-2)(0.5) = -\$0.50.

$E(X^2)$ = (\$$1^2$)(0.5) + (\$-2^2)(0.5) = (\$1)(0.5) + (\$4)(0.5) = \$2.50.

So $\sigma^2 = E(X^2) - \mu^2$ = \$2.50 - \$-0.50^2 = \$2.25. **σ = \$1.50**.

6. Bernoulli Random Variables, ch. 5.1.

If X = 1 with probability p, and X = 0 otherwise, then X = *Bernoulli (p)*.

Probability mass function (pmf):

$P(X = 1) = p$

$P(X = 0) = q$, where $p+q = 100\%$.

If X is Bernoulli (p), then μ = E(X) = p, and σ = √(pq).

For example, suppose X = 1 if you have a pocket pair next hand; X = 0 if not.

p = 5.88%. So, q = 94.12%.

[Two ways to figure out p:

(a) Out of choose(52,2) combinations for your two cards, 13 * choose(4,2) are pairs.

13 * choose(4,2) / choose(52,2) = 5.88%.

(b) Imagine *ordering* your 2 cards. No matter what your 1st card is, there are 51 equally likely choices for your 2nd card, and 3 of them give you a pocket pair. 3/51 = 5.88%.]

μ = E(X) = .0588. SD = σ = √(.0588 * 0.9412) = 0.235.

7. Moment generating functions, ch. 4.7

Suppose X is a random variable. $E(X)$, $E(X^2)$, $E(X^3)$, etc. are the *moments* of X.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Take derivatives with respect to t of $ø_X(t)$ and evaluate at t=0 to get moments of X.

1st derivative $(d/dt)\, e^{tX} = X\, e^{tX}$, $(d/dt)^2\, e^{tX} = X^2\, e^{tX}$, etc.

$(d/dt)^k\, E(e^{tX}) = E[(d/dt)^k\, e^{tX}] = E[X^k\, e^{tX}]$, (see p.84)

so $ø'_X(0) = E[X^1\, e^{0X}] = E(X)$,

$ø''_X(0) = E[X^2\, e^{0X}] = E(X^2)$, etc.

The moment gen. function $ø_X(t)$ uniquely characterizes the distribution of X.

So to show that X is, say, Bernoulli, you just need to show that it has the moment generating function of a Bernoulli random variable.

Also, if X_i are random variables with cdfs F_i, and $ø_{X_i}(t) \to ø(t)$, where $ø_X(t)$ is the moment generating function of X which has cdf F, then $X_i \to X$ in distribution, i.e. $F_i(y) \to F(y)$ for all y where F(y) is continuous, see p85.

7. Moment generating functions, continued.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Suppose X is Bernoulli (0.4). What is $ø_X(t)$?

$E(e^{tX}) = (0.6)\ (e^{t(0)}) + (0.4)\ (e^{t(1)}) = 0.6 + 0.4\ e^t$.

Suppose X is Bernoulli (0.4) and Y is Bernoulli (0.7) and X and Y are independent.

What is the distribution of XY?

$ø_{XY}(t) = E(e^{tXY}) = P(XY=0)\ (e^{t(0)}) + P(XY=1)(e^{t(1)})$

$= P(X=0 \text{ or } Y=0)\ (1) + P(X=1 \text{ and } Y=1)e^t$

$= [1 - P(X=1)P(Y=1)] + P(X=1)P(Y=1)e^t$

$= [1 - 0.4 \times 0.7] + 0.4 \times 0.7e^t$

$= 0.72 + 0.28e^t$, which is the moment generating function of a Bernoulli (0.28) random variable. Therefore XY is Bernoulli (0.28).

What about $Z = \min\{X,Y\}$?

Z = XY in this case, since X and Y are 0 or 1, so the answer is the same.

8. Independent random variables.

If X and Y are independent random variables, then

$E[f(X)\, g(Y)] = E[f(X)]\, E[g(Y)]$, for any functions f and g.

This is useful for problem 5.4 in hw2.

9. Binomial Random Variables, ch. 5.2.

Suppose now X = # of times something with prob. p occurs, out of n independent trials

Then X = *Binomial (n.p)*.

e.g. the number of pocket pairs, out of 10 hands.

Now X could = 0, 1, 2, 3, …, or n.

pmf: $P(X = k) = \text{choose}(n, k) * p^k q^{n-k}$.

e.g. say n=10, k=3: $P(X = 3) = \text{choose}(10,3) * p^3 q^7$.

Why? Could have 1 1 1 0 0 0 0 0 0 0, or 1 0 1 1 0 0 0 0 0 0, etc.

choose(10, 3) choices of places to put the 1's, and for each the prob. is $p^3 q^7$.

Key idea: $X = Y_1 + Y_2 + \ldots + Y_n$, where the Y_i are independent and *Bernoulli* (p).

If X is Bernoulli (p), then $\mu = p$, and $\sigma = \sqrt{(pq)}$.

If X is Binomial (n,p), then $\mu = np$, and $\sigma = \sqrt{(npq)}$.

9. Binomial Random Variables, continued.

Suppose X = the number of pocket pairs you get in the next 100 hands.

<u>What's P(X = 4)? What's E(X)? σ?</u> X = Binomial (100, 5.88%).

$P(X = k) = \text{choose}(n, k) * p^k q^{n-k}$.

So, $P(X = 4) = \text{choose}(100, 4) * 0.0588^4 * 0.9412^{96} = 13.9\%$, or 1 in **7.2.**

$E(X) = np = 100 * 0.0588 =$ **5.88**. $\sigma = \sqrt{(100 * 0.0588 * 0.9412)} =$ **2.35**.

So, out of 100 hands, you'd *typically* get about 5.88 pocket pairs, +/- around 2.35.