Stat 100a: Introduction to Probability.

Outline for the day:

1. Continuous random variables in general.
2. Exponential random variables.
3. Uniform, random variables.
4. Normal random variables.
5. Negative binomial random variables.
6. Moment generating functions of different distributions.
7. Survivor functions.
8. Violette/Elezra pot odds example.
9. P(4 of a kind) examples.
10. Heads up with KK.
11. Assign teams.

Read through chapter 6.4!

♣ ♥ ◆ ♠

1. Continuous random variables and their densities, p103-107.

Density (or pdf = Probability Density Function) f(y):

$$\int_B f(y)\, dy = P(X \text{ in } B).$$

Expected value, $\mu = E(X) = \int y\, f(y)\, dy$. (= $\sum y\, P(y)$ for discrete X.)

Variance, $\sigma^2 = V(X) = E(X^2) - \mu^2$.

$SD(X) = \sqrt{V(X)}$.

For examples of pdfs, see p104, 106, and 107.

2. Exponential distribution, ch 6.4.

Useful for modeling waiting times til something happens (like the geometric).

pdf of an exponential random variable is $f(y) = \lambda\ exp(-\lambda\ y)$, for $y \geq 0$, and $f(y) = 0$ otherwise.
If X is exponential with parameter λ, then $E(X) = SD(X) = 1/\lambda$

If the total numbers of events in any disjoint time spans are independent, then these totals are Poisson random variables. If in addition the events are occurring at a constant rate λ, then the times between events, or *interevent times*, are exponential random variables with mean $1/\lambda$.

Example. Suppose you play 20 hands an hour, with each hand lasting exactly 3 minutes, and let X be the time in hours until the end of the first hand in which you are dealt pocket aces. Use the exponential distribution to approximate $P(X \leq 2)$ and compare with the exact solution using the geometric distribution.

Answer. Each hand takes 1/20 hours, and the probability of being dealt pocket aces on a particular hand is 1/221, so the rate λ = 1 in 221 hands = 1/(221/20) hours ~ 0.0905 per hour.
Using the exponential model, $P(X \le 2 \text{ hours}) = 1 - \exp(-2\lambda) \sim 16.556\%$.
This is an approximation, however, since by assumption X is not continuous but must be an integer multiple of 3 minutes.
Let Y = the number of hands you play until you are dealt pocket aces. Using the geometric distribution, $P(X \le 2 \text{ hours}) = P(Y \le 40 \text{ hands})$
$= 1 - (220/221)^{40} \sim 16.590\%$.

The survivor function for an exponential random variable is particularly simple: $P(X > c) = \int_c^\infty f(y)dy = \int_c^\infty \lambda \exp(-\lambda y)dy = -\exp(-\lambda y)]_c^\infty = \exp(-\lambda c)$.

Like geometric random variables, exponential random variables have the *memorylessness* property: if X is exponential, then for any non-negative values a and b, $P(X > a+b \mid X > a) = P(X > b)$.
Thus, with an exponential (or geometric) random variable, if after a certain time you still have not observed the event you are waiting for, then the distribution of the *future,* additional waiting time until you observe the event is the same as the distribution of the *unconditional* time to observe the event to begin with.

3. Uniform Random Variables and R.

Continuous random variables are often characterized by their

probability density functions (pdf, or *density*): a function f(x)

such that P{X is in B} = $\int_B f(x)\,dx$.

Uniform: f(x) = c, for x in (a, b).

= 0, for all other x.

[Note: c must = 1/(b-a), so that $\int_a^b f(x)\,dx$ = P{X is in (a,b)} = 1.]

Uniform (0,1). See p107-109.

f(y) = 1, for y in (0,1). $\mu = 0.5$. $\sigma \sim 0.29$.

P(X is between 0.4 and 0.6) = $\int_{.4}^{.6} f(y)\,dy = \int_{.4}^{.6} 1\,dy = 0.2$.

In R, runif(1,min=a,max=b) produces a pseudo-random uniform.

Uniform example.

For a continuous random variable X,
The pdf f(y) is a function where $\int_a^b f(y)dy = P\{X \text{ is in } (a,b)\}$,
$E(X) = \mu = \int_{-\infty}^{\infty} y\, f(y)dy$,
and $\sigma^2 = Var(X) = E(X^2) - \mu^2$. $sd(X) = \sigma$.

For example, suppose X and Y are independent uniform random variables on (0,1), and $Z = min(X,Y)$. **a)** Find the pdf of Z. **b)** Find E(Z). **c)** Find SD(Z).

a. For c in (0,1), $P(Z > c) = P(X > c \ \&\ Y > c) = P(X > c)\ P(Y > c) = (1-c)^2 = 1 - 2c + c^2$.
So, $P(Z \le c) = 1 - (1 - 2c + c^2) = 2c - c^2$.
Thus, $\int_0^c f(c)dc = 2c - c^2$. So f(c) = the derivative of $2c - c^2 = 2 - 2c$, for c in (0,1).
Obviously, f(c) = 0 for all other c.
b. $E(Z) = \int_{-\infty}^{\infty} y\, f(y)dy = \int_0^1 c\ (2-2c)\ dc = \int_0^1 2c - 2c^2\ dc = c^2 - 2c^3/3]_{c=0}^{1}$
$= 1 - 2/3 - (0 - 0) = 1/3$.
c. $E(Z^2) = \int_{-\infty}^{\infty} y^2\, f(y)dy = \int_0^1 c^2\ (2-2c)\ dc = \int_0^1 2c^2 - 2c^3\ dc = 2c^3/3 - 2c^4/4]_{c=0}^{1}$
$= 2/3 - 1/2 - (0 - 0) = 1/6$.
So, $\sigma^2 = Var(Z) = E(Z^2) - [E(Z)]^2 = 1/6 - (1/3)^2 = 1/18$.
$SD(Z) = \sigma = \sqrt{(1/18)} \sim 0.2357$.

4. Normal random variables.

So far we have seen two continuous random variables, the uniform and the exponential.

Normal. pp 115-117. mean = μ, SD = σ, $f(y) = 1/\sqrt{(2\pi\sigma^2)}\ e^{-(y-\mu)^2/2\sigma^2}$.

Symmetric around μ,

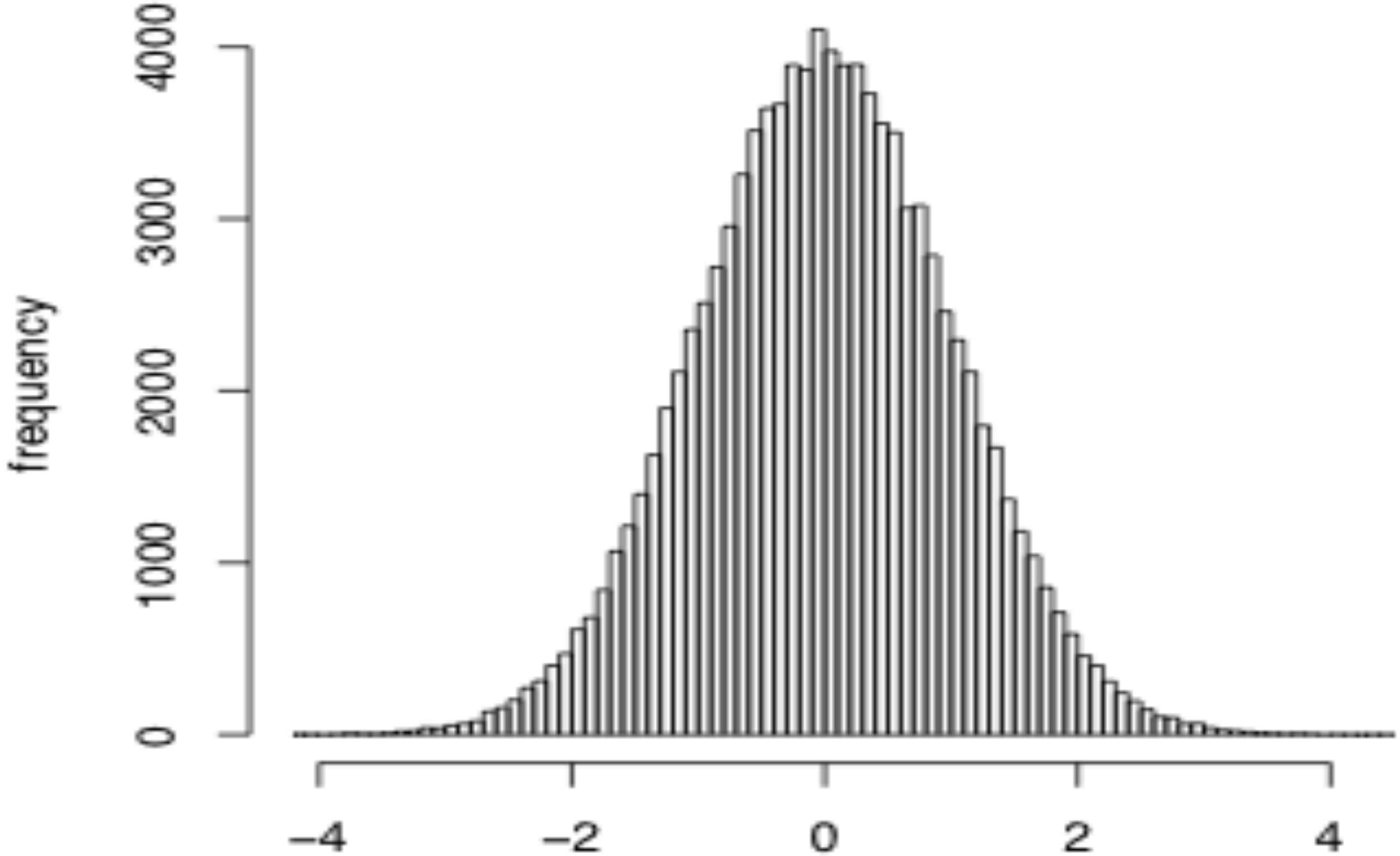
50% of the values are within 0.674 SDs of μ,

68.27% of the values are within 1 SD of μ, and

95% are within 1.96 SDs of μ.

* Standard Normal. Normal with $\mu = 0$, $\sigma = 1$. See pp 117-118.

Standard normal density:
68.27% between -1.0 and 1.0
95% between -1.96 and 1.96



5. Negative Binomial Random Variables, ch 5.4.

Recall: if each trial is independent, and each time the probability of an occurrence is p, and X = # of trials until the *first* occurrence, then:

X is Geometric (p), $P(X = k) = p^1 q^{k-1}$, $\mu = 1/p$, **σ** $= (\sqrt{q}) \div p$.

Suppose now X = # of trials until the *rth* occurrence.

Then X = *negative binomial (r,p)*.

e.g. the number of hands you have to play til you've gotten r=3 pocket pairs.

Now X could be 3, 4, 5, …, up to ∞.

pmf: $P(X = k) = \text{choose}(k-1, r-1)\ p^r q^{k-r}$, for k = r, r+1, ….

e.g. say r=3 & k=7: $P(X = 7) = \text{choose}(6,2)\ p^3 q^4$.

Why? Out of the first 6 hands, there must be exactly r-1 = 2 pairs. Then pair on 7th.

P(exactly 2 pairs on first 6 hands) = $\text{choose}(6,2)\ p^2 q^4$. P(pair on 7th) = p.

If X is negative binomial (r,p), then $\mu = r/p$, and $\sigma = (\sqrt{rq}) \div p$.

e.g. Suppose X = the number of hands til your 12th pocket pair. P(X = 100)? E(X)? **σ**?

X = Neg. binomial (12, 5.88%).

$P(X = 100) = \text{choose}(99,11)\ p^{12} q^{88}$

= choose(99,11) * 0.0588 ^ 12 * 0.9412 ^ 88 = **0.104%.**

E(X) = r/p = 12/0.0588 ~ **204**. **σ** = sqrt(12*0.9412) / 0.0588 = **57.2**.

So, you'd typically *expect* it to take 204 hands til your 12th pair, +/- around 57.2 hands.

6. Moment generating functions of some random variables.

Bernoulli(p). $ø_X(t) = pe^t + q$.

Binomial(n,p). $ø_X(t) = (pe^t + q)^n$. p94.

Geometric(p). $ø_X(t) = pe^t/(1 - qe^t)$.

Neg. binomial (r,p). $ø_X(t) = [pe^t/(1 - qe^t)]^r$. p97.

Poisson(λ). $ø_X(t) = e^{\{\lambda e^t - \lambda\}}$. p100.

Uniform (a,b). $ø_X(t) = (e^{tb} - e^{ta})/[t(b-a)]$. p108.

Exponential (λ). $ø_X(t) = \lambda/(\lambda - t)$. p123.

Normal. $ø_X(t) = e^{\{t\mu + t^2\sigma^2/2\}}$.

Note t is missing in neg. binomial one on p97.

7. Survivor functions. p96 and 115.

Recall the cdf $F(b) = P(X \le b)$.
The survivor function is $S(b) = P(X > b) = 1 - F(b)$.
Some random variables have really simple survivor functions and it can be convenient to work with them.
If X is geometric, then $S(b) = P(X > b) = q^b$, for $b = 0,1,2,\ldots$.
For instance, let b=2. X > 2 means the 1st two were misses,
i.e. $P(X>2) = q^2$.
For exponential X, $F(b) = 1 - exp(-\lambda b)$, so $S(b) = exp(-\lambda b)$.

An interesting fact is that, if X takes on only values in $\{0,1,2,3,\ldots\}$,
then $E(X) = S(0) + S(1) + S(2) + \ldots$.
Proof. See p96.

$$
\begin{aligned}
S(0) &= P(X=1) + P(X=2) + P(X=3) + P(X=4) + \ldots \\
S(1) &= \phantom{P(X=1) + {}} P(X=2) + P(X=3) + P(X=4) + \ldots \\
S(2) &= \phantom{P(X=1) + P(X=2) + {}} P(X=3) + P(X=4) + \ldots \\
S(3) &= \phantom{P(X=1) + P(X=2) + P(X=3) + {}} P(X=4) + \ldots
\end{aligned}
$$

Add these up and you get
$0\,P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) + 4P(X=4) + \ldots$
$= \sum kP(X=k) = E(X)$.

8. Pot odds ex. *Poker Superstars Invitational Tournament*, FSN, October 2005.

Ted Forrest: 1 million chips
Freddy Deeb: 825,000 Blinds: 15,000 / 30,000
Cindy Violette: 650,000
Eli Elezra: 575,000

* Elezra raises to 100,000
* Forrest folds.
* Deeb, the small blind, folds.
* Violette, the big blind with K♦ J♦, calls.

* The flop is: 2♦ 7♣ A♦

* Violette bets 100,000. (pot = 315,000).
* Elezra raises all-in to 475,000. (pot = 790,000).

So, it's 375,000 more to Violette. She folds.

Q: Based on expected value, should she have called?
Her chances must be at least 375,000 / (790,000 + 375,000) = 32%.

Violette has K♦ J♦.   The flop is: 2♦ 7♣ A♦.

Q: Based on expected value, should she have called?

Her chances must be at least 375,000 / (790,000 + 375,000) = 32%.

vs.	AQ: 38%.	AK: 37%	AA: 26%	77: 26%	A7: 31%
	A2: 34%	72: 34%	TT: 54%	T9: 87%	73: 50%

Harrington's principle: always assume at least a 10% chance that opponent is bluffing.

Bayesian approach: average all possibilities, weighting them by their likelihood.

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Harrington's principle: always assume at least a 10% chance that opponent is bluffing.
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Reality: Elezra had 7♦ 3♥. Her chances were 51%. Bad fold.
What was her prob. of winning (given just her cards and Elezra's, and the flop)?
Of choose(45,2) = 990 combinations for the turn & river, how many give her the win?
First, how many outs did she have? eight ♦s + 3 kings + 3 jacks = 14.
She wins with (out, out) or (out, nonout) or (non-♦ Q, non-♦ T)
choose(14,2) + 14 x 31 + 3 * 3 = 534
but not (k or j, 7 or non-♦ 3) and not (3♦ , 7 or non-♦ 3)
- 6 * 4 - 1 * 4 = 506.
So the answer is 506 / 990 = 51.1%.

9. 4 of a kind examples.

Suppose you're all in next hand, no matter what cards you get.

P(eventually make 4-of-a-kind)? [including case where all 4 are on board]

Trick: just forget card order, and consider all collections of 7 cards.

Out of choose(52,7) different combinations, each equally likely, how many of them involve 4-of-a-kind?

13 choices for the 4-of-a-kind.

For each such choice, there are choose(48,3) possibilities for the other 3 cards.

So, P(4-of-a-kind) = 13 * choose(48,3) / choose(52,7) ~ 0.168%, or 1 in **595.**

P(flop 4-of-a-kind) =

13*48 / choose(52,5) = 0.024% = 1 in **4165**.

P(flop 4-of-a-kind | pocket pair)?

No matter which pocket pair you have, there are choose(50,3) possible flops, each equally likely, and how many of them give you 4-of-a-kind?

48. (e.g. if you have 7♣ 7♥, then need to flop 7♦ 7♠ x, & there are 48 choices for x)

So P(flop 4-of-a-kind | pp) = 48/choose(50,3) = 0.245% = 1 in **408**.

10. Heads up with KK?

Recall Dan Harrington says that,

"with a hand like AA, you really want to be one-on-one."

It's questionable though.

<u>Do you want multiple callers with KK?</u>

a) You have \$100 and KK and are all-in against TT. You're 81% to double up, so your expected number of chips after the hand is

$0.81 \times \$200 = \mathbf{\$162}$.

b) You have \$100 and KK and are all-in against A9 and TT. You're 58% to have \$300, so your expected value is **\$174**.

- So, if you have KK and an opponent with TT has already called you, and another who has A9 is thinking about whether to call you too, you actually want A9 to call!
- Given this, one may question Harrington's suggested strategy of raising huge in order to isolate yourself against one player.

11. Project teams.

The project is problem 8.2, p165.

You need to write code to go all in or fold. In R, try:

```
install.packages(holdem)
library(holdem)
library(help="holdem")
```

timemachine, tommy, ursula, vera, william, and xena are examples.

crds1[1,1] is your higher card (2-14).

crds1[2,1] is your lower card (2-14).

crds1[1,2] and crds1[2,2] are suits of your higher card & lower card.

```
help(tommy)
tommy
```

```
function (numattable1, crds1, board1, round1, currentbet, mychips1,
  pot1, roundbets, blinds1, chips1, ind1, dealer1, tablesleft)
{ a1 = 0
  if (crds1[1, 1] == crds1[2, 1])
     a1 = mychips1
  a1
}
```

help(vera)

All in with a pair, any suited cards, or if the smaller card is at least 9.

```
function (numattable1, crds1, board1, round1, currentbet, mychips1,
    pot1, roundbets, blinds1, chips1, ind1, dealer1, tablesleft)
{a1 = 0
 if ((crds1[1, 1] == crds1[2, 1]) || (crds1[1, 2] == crds1[2,2]) ||
     (crds1[2, 1] > 8.5)) a1 = mychips1
   a1
}
```

You need to email me your function, to frederic@stat.ucla.edu. It should be written (or cut and pasted) simply into the body of the email. If you write it in Word, save as text first, and then paste it into the email. Just submit one email per team.

For instance, if your letter is “b”, you might do:

For instance, if your letter is “b”, you might do:

```
bruin = function (numattable1, crds1, board1, round1, currentbet,
      mychips1, pot1, roundbets, blinds1, chips1, ind1, dealer1,
      tablesleft) {
## all in with any pair higher than 7s, or if lower card is J or higher
a1 = 0
if ((crds1[1, 1] == crds1[2, 1]) && (crds1[1, 1] > 6.5)) a1 = mychips1
if (crds1[2,1] > 10.5) a1 = mychips1
a1
} ## end of bruin
```