Stat 100a: Introduction to Probability.

Outline for the day:

1. Multiple runs, expected value and variance.
2. Rainbow flops.
3. HW3 problems.
4. P(flop an ace high flush), P(flop a straight flush), etc.
5. Review list.
6. Cards til 2^{nd} king.
7. P(2 pairs).
8. P(suited King).
9. Hellmuth vs. Farha and P(flush).

Mid 1 is Thu Feb 5.
For project teams see teamsresults.txt.
For emails see roster100a.txt today Tue Feb 3 1-3pm.
Project is due Tue Mar 3 8pm by email.

1. Multiple runs, expected value and variance.

E(X+Y) = E(X) + E(Y). *Whether X & Y are independent or not*!

Similarly, E(X + Y + Z + …) = E(X) + E(Y) + E(Z) + …

And, if X & Y are independent, then V(X+Y) = V(X) + V(Y).

so SD(X+Y) = √[SD(X)^2 + SD(Y)^2].

Also, if Y = 9X, then E(Y) = 9E(X), and SD(Y) = 9SD(X). V(Y) = 81V(X).

Farha vs. Antonius.

Running it 4 times. Let X = chips you have after the hand. Let p be the prob. you win.

$X = X_1 + X_2 + X_3 + X_4$, where X_1 = chips won from the first "run", etc.

$E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$

= 1/4 pot (p) + 1/4 pot (p) + 1/4 pot (p) + 1/4 pot (p)

= pot (p)

= same as E(Y), where Y = chips you have after the hand if you ran it once!

But the SD is smaller: clearly $X_1 = Y/4$, so $SD(X_1) = SD(Y)/4$. So, $V(X_1) = V(Y)/16$.

$V(X) \sim V(X_1) + V(X_2) + V(X_3) + V(X_4)$,

$= 4\ V(X_1)$

$= 4\ V(Y) / 16$

$= V(Y) / 4$. **So SD(X) = SD(Y) / 2.**

2. Rainbow flops.

P(Rainbow flop) = choose(4,3) * 13 * 13 * 13 ÷ choose(52,3)

choices for the 3 suits — numbers on the 3 cards — possible flops

~ 39.76%.

Q: Out of 100 hands, what is the expected number of rainbow flops? +/- what?

X = Binomial (n,p), with n = 100, p = 39.76%, q = 60.24%.

E(X) = np = 100 * 0.3976 = 39.76

SD(X) = √(npq) = sqrt(23.95) = 4.89.

So, expect around 39.76 +/- 4.89 rainbow flops, out of 100 hands.

Rainbow flops, continued.

P(Rainbow flop) ~ 39.76%.

Q: Let X = the number of hands til your 4^{th} rainbow flop.

What is P(X = 10)? What is E(X)? What is SD(X)?

X = negative binomial (r,p), with r = 4, p = 39.76%, q = 60.24%.

$P(X = k) = \text{choose}(k-1, r-1)\ p^r\ q^{k-r}$.

Here k = 10. $P(X = 10) = \text{choose}(9,3)\ 39.76\%^4\ 60.24\%^6 = 10.03\%$.

$\mu = E(X) = r/p = 4 \div 0.3976 = 10.06$ hands.

$\sigma = SD(X) = (\sqrt{rq}) / p =$ sqrt(4*0.6024) / 0.3976 = 3.90 hands.

So, you expect it *typically* to take around

10.06 +/- 3.90 hands til your 4th rainbow flop.

3. Hw3 problems.

6.12 and 7.14b are hard.

For 6.12, there are different ways to do this problem, but one way leads you to some tough integrals. You can use the fact that

$\int c\,[2\exp(-2c)]\,dc$ is the expected value of an exponential random variable with parameter $\lambda = 2$, and this expected value is 1/2. Similarly, $\int c\,\exp(-c)\,dc$ is the expected value of an exponential with $\lambda = 1$, which is 1.

For 7.14b, let $x = r^2$, and you'll get an equation like $-x^3 + 2x - 1 = 0$,

so $(x-1)(-x^2 - x + 1) = 0$. There are 3 possible solutions to this:

$x = 1$, or $x^2 + x - 1 = 0$, so $x = [-1 \pm \sqrt{(1+4)}]/2$. Now reason why we can rule out two of these possibilities for x.

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4. P(flop an ace high flush)? [where the ace might be on the board]

-- 4 suits

-- one of the cards must be an ace. choose(12,4) possibilities for the others.

So P(flop ace high flush) = 4 * choose(12,4) / choose(52,5)

= 0.0762%, or 1 in **1313**.

P(flop a straight flush)?

-- 4 suits

-- 10 different straight-flushes in each suit. (5 high, 6 high, …, Ace high)

So P(flop straight flush) = 4 * 10 / choose(52,5)

= 0.00154%, or 1 in **64974**.

5. <u>Review</u> List. We have basically done up through ch7.1.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. $P(AB) = P(A)\,P(B|A)$ [$= P(A)P(B)$ if ind.]
7) Odds ratios.
8) Random variables (RVs).
9) Discrete and continuous RVs, pmf, cdf, pdf, survivor function.
10) Expected value, variance, and SD.
11) Pot odds calculations.
12) Bayes's rule.
13) Bernoulli RV. [0-1. $\mu = p$, $\sigma = \sqrt{(pq)}$.]
14) Binomial RV. [# of successes, out of n tries. $\mu = np$, $\sigma = \sqrt{(npq)}$.]
15) Geometric RV. [# of tries til 1st success. $\mu = 1/p$, $\sigma = (\sqrt{q}) / p$.]
16) Negative binomial RV. [# of tries til rth success. $\mu = r/p$, $\sigma = (\sqrt{rq}) / p$.]
17) Poisson RV. [$\mu = \lambda$, $\sigma = \sqrt{\lambda}$.]
18) E(X+Y), V(X+Y), and E[f(X) g(Y)] when X and Y are ind.
19) Moment generating functions.
20) Uniform, exponential, and normal random variables.
21) Markov and Chebyshev inequalities.

5. Review List. We have basically done up through ch7.1.

For this midterm, ignore Sections 4.4, 6.6, 6.7, and most of 6.3 which is about optimal strategy in simplified versions of poker. You do need to know the basics of the uniform distribution though, so just the 1st page or so of ch 6.3.

6. Cards til 2^{nd} king.

Deal the cards face up, without reshuffling.

Let Z = the number of cards til the 2nd king.

What is E(Z)?

This is a really hard question.

The solution uses the fact that

$$E(X+Y+Z + ...) = E(X) + E(Y) + E(Z) + ...$$

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E(cards til 2^{nd} king).

Z = the number of cards til the 2nd king. What is E(Z)?

Let X_1 = number of non-king cards before 1^{st} king.

Let X_2 = number of non-kings after 1^{st} king til 2^{nd} king.

Let X_3 = number of non-kings after 2^{nd} king til 3^{rd} king.

Let X_4 = number of non-kings after 3^{rd} king til 4^{th} king.

Let X_5 = number of non-kings after 4^{th} king til the end of the deck.

Clearly, $X_1 + X_2 + X_3 + X_4 + X_5 + 4 = 52$.

By symmetry, $E(X_1) = E(X_2) = E(X_3) = E(X_4) = E(X_5)$.

Therefore, $E(X_1) = E(X_2) = 48/5$.

$Z = X_1+X_2+2$, so $E(Z) = E(X_1) + E(X_2) + 2 = 48/5 + 48/5 + 2 = 21.2$.

7. P(flop two pairs).

If you’re sure to be all-in next hand, what is P(you will flop two pairs)?

This is a tricky one. Don’t double-count (4♣ 4♥ 9♣ 9♥ Q♣) and (9♣ 9♥ 4♣ 4♥ Q♣)!

There are choose(13,2) possibilities for the NUMBERS of the two pairs.

For each such choice (such as 4 & 9),

there are choose(4,2) choices for the suits of the lower pair,

and same for the suits of the higher pair.

So, choose(13,2) * choose(4,2) * choose(4,2) different possibilities for the two pairs.

For each such choice, there are 44 [52 - 8 = 44] different possibilities for your fifth card, so that it’s not a full house but simply two pairs. So,

P(flop two pairs) = choose(13,2) * choose(4,2) * choose(4,2) * 44 / choose(52,5)

~ 4.75%, or 1 in **21**.

8. What is the probability that you will be dealt a king and another card of the same suit as the king?

4 * 12 / C(52,2) = 3.62%.

The typical number of hands until this occurs is ...

1/.0362 ~ 27.6.
(√96.38%) / 3.62% ~ 27.1.
So the answer is 27.6 +/- 27.1.

9. Hellmuth vs. Farha.

P(Hellmuth makes a flush)

$$= \frac{C(11,5) + C(11,4) * 37 + C(11,3) * C(37,2)}{C(48,5)} = 7.16\%.$$

P(Farha makes a flush)

$$= \frac{2 * (C(12,5) + C(12,4) * 36)}{C(48,5)} = 2.17\%.$$