Stat 100a: Introduction to Probability.

Outline for the day:

1. Random walks.
2. Reflection prionciple.
3. Ballot theorem.
4. Avoiding zero.

Remember hw2 is due Thu Feb 19.

Read 4.4 and 6.3 for next Tue Feb 24.

Exam 2 is Tue Mar 3.

The project is due Tue Mar 3 8pm by email to me.

NO CLASS or OH Tue Mar 10.

Hw3 is due Mar 12.

Exam 3 is Thu Mar 12.

1. Random walks, ch. 7.6**.**

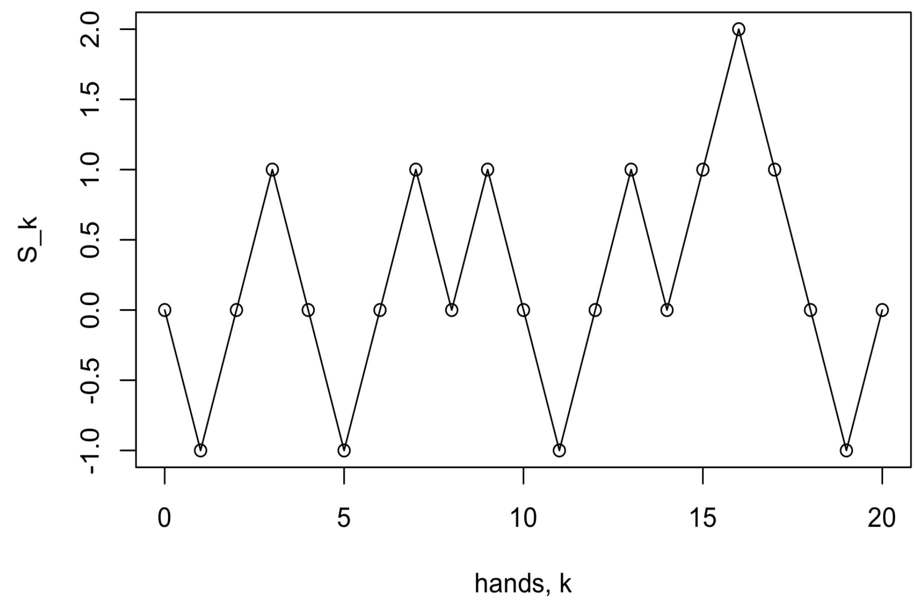
Suppose that $X_1, X_2, \ldots$, are iid,

and $S_k = X_0 + X_1 + \ldots + X_k$ for $k = 0, 1, 2, \ldots$.

The totals $\{S_0, S_1, S_2, \ldots\}$ form a *random walk*.

The classical *(simple)* case is when each X_i is

1 or -1 with probability ½ each.



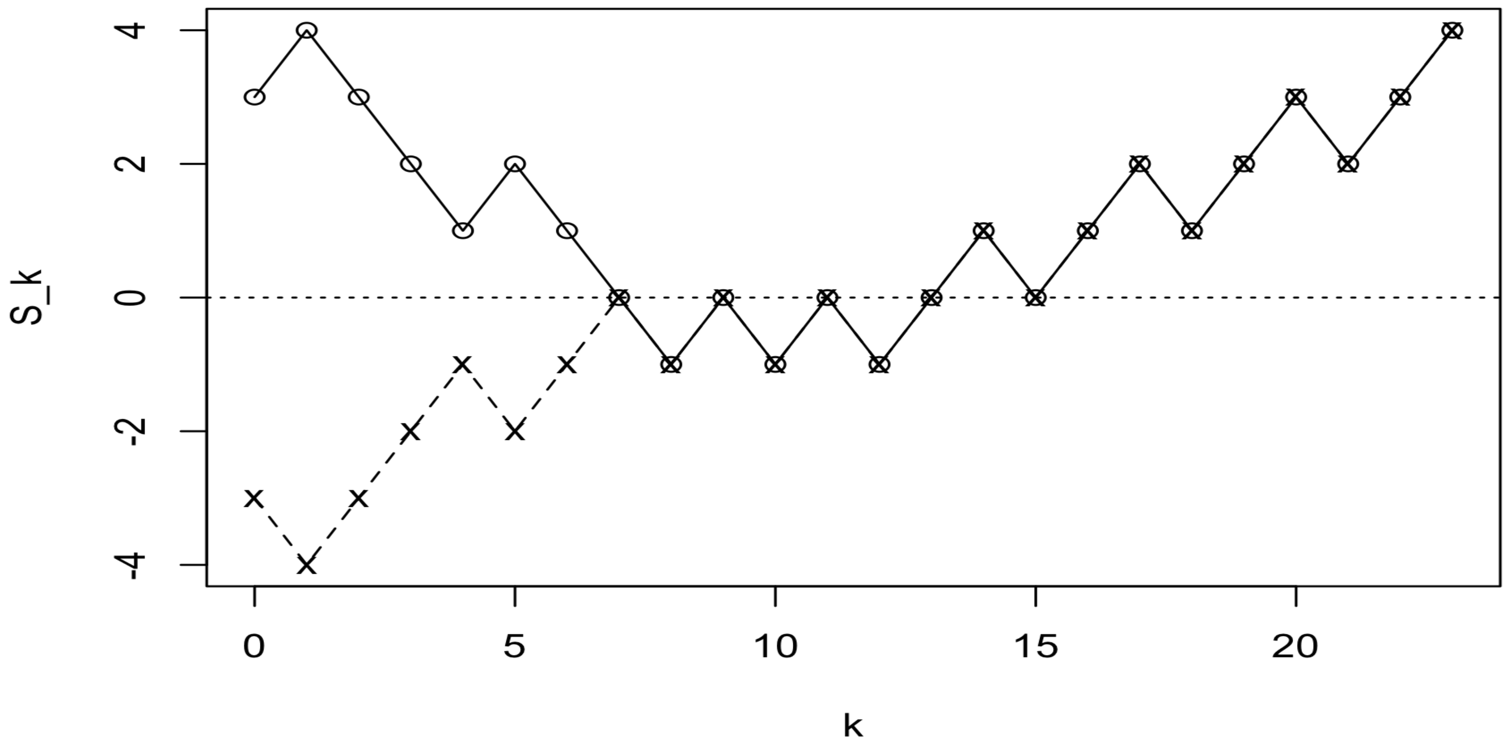
* *Reflection principle:* The number of paths from $(0,X_0)$ to (n,y) that touch the x-axis = the number of paths from $(0,-X_0)$ to (n,y), for any n,y, and $X_0 > 0$.

* *Ballot theorem:* In $n = a+b$ hands, if player A won a hands and B won b hands, where $a>b$, and if the hands are aired in random order, P(A won more hands than B *throughout* the telecast) $= (a-b)/n$.

[In an election, if candidate X gets x votes, and candidate Y gets y votes, where $x > y$, then the probability that X always leads Y throughout the counting is $(x-y) / (x+y)$.]

* For a simple random walk, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$, for any even n.

2. Reflection Principle. The number of paths from $(0,X_0)$ to (n,y) that touch the x-axis = the number of paths from $(0,-X_0)$ to (n,y), for any n,y, and $X_0 > 0$.



For each path from $(0,X_0)$ to (n,y) that touches the x-axis, you can reflect the first part til it touches the x-axis, to find a path from $(0,-X_0)$ to (n,y), and vice versa.

Total number of paths from $(0,-X_0)$ to (n,y) is easy to count: it's just C(n,a), where you go up *a* times and down *b* times

[i.e. $a-b = y - (-X_0) = y + X_0$. $a+b=n$, so $b = n-a$, $2a-n=y+X_0$, $a=(n+y+X_0)/2$].

3. Ballot theorem. In n = a+b hands, if player A won a hands and B won b hands, where a>b, and if the hands are aired in random order,

then P(A won more hands than B *throughout* the telecast) = (a-b)/n.

Proof: We know that, after n = a+b hands, the total difference in hands won is a-b.

Let x = a-b.

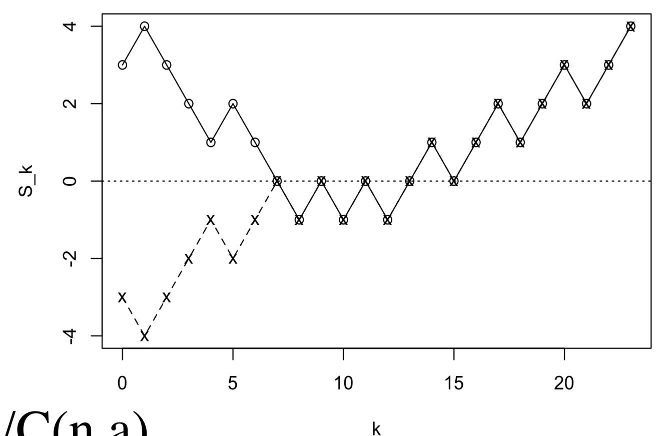
We want to count the number of paths from (1,1) to (n,x) that do not touch the x-axis.
By the reflection principle, the number of paths from (1,1) to (n,x) that **do** touch the x-axis equals the total number of paths from (1,-1) to (n,x).
So the number of paths from (1,1) to (n,x) that **do not** touch the x-axis equals the number of paths from (1,1) to (n,x) minus the number of paths from (1,-1) to (n,x)

= C(n-1,a-1) – C(n-1,a)

= (n-1)! / [(a-1)! (n-a)!] – (n-1)! / [a! (n-a-1)!]

= {n! / [a! (n-a)!]} [(a/n) - (n-a)/n]

= C(n,a) (a-b)/n.



And each path is equally likely, and has probability 1/C(n,a).

So, P(going from (0,0) to (n,a) without touching the x-axis = (a-b)/n.

4. Avoiding zero.

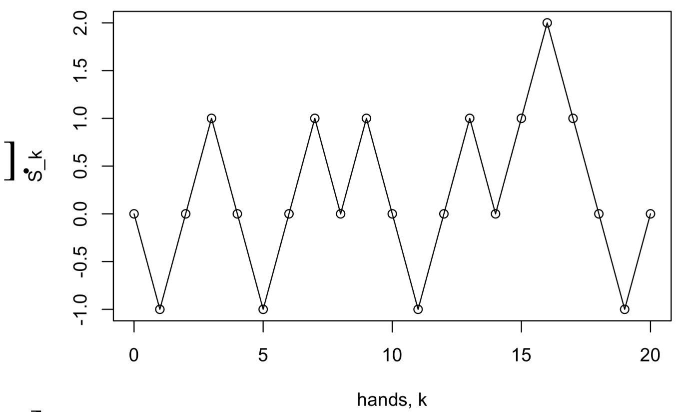
For a simple random walk, for any even # n, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.

Proof. The number of paths from (0,0) to (n, j) that don't touch the x-axis at positive times

= the number of paths from (1,1) to (n,j) that don't touch the x-axis at positive times

= paths from (1,1) to (n,j) - paths from (1,-1) to (n,j) by the *reflection principle*

$= N_{n-1,j-1} - N_{n-1,j+1}$



Let $Q_{n,j} = P(S_n = j)$.

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = j) = ½[Q_{n-1,j-1} - Q_{n-1,j+1}]$.

Summing from $j = 2$ to ∞,

$P(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n > 0)$

$= ½[Q_{n-1,1} - Q_{n-1,3}] + ½[Q_{n-1,3} - Q_{n-1,5}] + ½[Q_{n-1,5} - Q_{n-1,7}] + \ldots$

$= (1/2)\, Q_{n-1,1}$

$= (1/2)\, P(S_n = 0)$, because to end up at (n, 0), you have to be at (n-1,+/-1),

so $P(S_n = 0) = (1/2)\, Q_{n-1,1} + (1/2)\, Q_{n-1,-1} = Q_{n-1,1}$.

By the same argument, $P(S_1 < 0, S_2 < 0, \ldots, S_{n-1} < 0, S_n < 0) = (1/2)\, P(S_n = 0)$.

So, $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_n = 0)$.