Stat 100a: Introduction to Probability.

Outline for the day:

1. Review list.
2. Random walk example.
3. Bayes' rule example.
4. Conditional probability examples.
5. Another luck and skill example.
6. Another random walk example.

Exam 2 is Tue Mar 3.
The project is due Tue Mar 3 8pm by email to me.
NO CLASS or OH Tue Mar 10.
Hw3 is due Mar 12.
Exam 3 is Thu Mar 12.
They are open book, open note.
Bring a calculator and a dark pen or dark pencil.

1. Review list.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. $P(AB) = P(A)\ P(B|A)$ [= $P(A)P(B)$ if ind.]
7) Odds ratios.
8) Random variables (RVs).
9) Discrete RVs, and probability mass function (pmf).
10) Expected value.
11) Pot odds calculations.
12) Luck, skill, and deal-making.
13) Variance and SD.
14) Bernoulli RV. [0-1. $\mu = p$, $\sigma = \sqrt{(pq)}$.]
15) Binomial RV. [# of successes, out of n tries. $\mu = np$, $\sigma = \sqrt{(npq)}$.]
16) Geometric RV. [# of tries til 1st success. $\mu = 1/p$, $\sigma = (\sqrt{q}) / p$.]

17) Negative binomial RV. [# of tries til rth success. μ = r/p, σ = (√rq) / p.]
18) Poisson RV [# of successes in some time interval. [μ = λ, σ = √λ.]
19) E(X+Y), V(X+Y) (ch. 7.1).
20) Bayes's rule (ch. 3.4).
21) Continuous RVs and probability density function (pdf)
22) Uniform RV
23) Normal RV
24) Exponential RVs.
25) Pareto RVs
26) Moment generating functions
27) Markov and Chebyshev inequalities.
28) Law of Large Numbers (LLN)
29) Central Limit Theorem (CLT)
30) Conditional expectation.
31) Confidence intervals for the sample mean.
32) Fundamental theorem of poker
33) Random Walks, Reflection Principle, Ballot Theorem, avoiding zero
34) Chip proportions and induction.

Basically, we've done all of ch. 1-7 except 6.7.

2. Another random walk example.

Suppose that a \$10 winner-take-all tournament has $32 = 2^5$ players. So, you need to double up 5 times to win. Winner gets \$320.

Suppose that on each hand of the tournament, you have probability $p = 0.7$ to double up, and with probability $q = 0.3$ you will be eliminated. What is your expected profit in the tournament?

Your expected *return* = (\$320) x P(win the tournament) + (\$0) x P(you don't win)

= (\$320) x 0.7^5 = \$53.78. But it costs \$10.

So expected *profit* = \$53.78 - \$10 = \$43.78.

Alternatively, your profit could = \$310 or \$-10, so

expected profit = (\$310)(0.7^5) + (-\$10)(1 - 0.7^5)

= \$52.10 - \$10 + \$1.68

= \$43.78.

3. Bayes' rule example.

Your opponent raises all-in before the flop. Suppose you think she would do that *80%* of the time with AA, KK, or QQ, and she would do that *30%* of the time with AK or AQ, and 1% of the time with anything else.

Given only this, and not even your cards, what's P(she has AK)?

Given nothing, P(AK) = 16/C(52,2) = 16/1326. P(AA) = C(4,2)/C(52,2) = 6/1326.

Using Bayes' rule,

$$P(AK \mid \text{all-in}) = \frac{P(\text{all-in} \mid AK) * P(AK)}{P(\text{all-in}|AK)P(AK) + P(\text{all-in}|AA)P(AA) + P(\text{all-in}|KK)P(KK) + \ldots},$$

$$= \frac{30\% \times 16/1326}{\underset{(AK)}{[30\%\text{x}16/1326]} + \underset{(AA)}{[80\%\text{x}6/1326]} + \underset{(KK)}{[80\%\text{x}6/1326]} + \underset{(QQ)}{[80\%\text{x}6/1326]} + \underset{(AQ)}{[30\%\text{x}16/1326]} + \underset{(\text{anything else})}{[1\%\ (1326-16-6-6-6-16)/1326)]}}.$$

= **13.06%.** Compare with 16/1326 ~ 1.21%.

4. Conditional prob. examples.

<u>Approximate</u> P(SOMEONE has AA, given you have KK)? Out of your 8 opponents?

Note that given that you have KK,

P(player 2 has AA & player 3 has AA)

= P(player 2 has AA) x P(player 3 has AA | player 2 has AA)

= choose(4,2) / choose(50,2) x 1/choose(48,2)

= 0.0000043, or 1 in 230,000.

So, very little overlap! Given you have KK,

P(someone has AA) = P(player2 has AA or player3 has AA or … or pl.9 has AA)

~ P(player2 has AA) + P(player3 has AA) + … + P(player9 has AA)

= 8 x choose(4,2) / choose(50,2) = 3.9%, or 1 in **26.**

What is **exactly** P(SOMEONE has an Ace | you have KK)? (8 opponents)

(or more than one ace)

Given that you have KK,

P(SOMEONE has an Ace) = 100% - P(nobody has an Ace).

And P(nobody has an Ace) = choose(46,16)/choose(50,16)

= **20.1%.**

So P(SOMEONE has an Ace) = **79.9%.**

Here are some more conditional probability examples.

P(You have AK | you have exactly one ace)?

= P(You have AK and exactly one ace) / P(exactly one ace)

= P(AK) / P(exactly one ace)

= (16/C(52,2)) ÷ (4x48/C(52,2))

= 4/48 = 8.33%.

P(You have AK | you have at least one ace)?

= P(You have AK and at least one ace) / P(at least one ace)

= P(AK) / P(at least one ace)

= 16/C(52,2) ÷ {4x48 + C(4,2)}/C(52,2) ~ 8.08%.

P(You have AK | your FIRST card is an ace)?

= 4/51 = 7.84%.

5. Another luck and skill example.

Hand 1 of Lederer vs. Minieri.

Minieri's increase in equity due to luck

= his expected number of chips after the cards are dealt - his expected number of chips before cards are dealt.

Let a = the number of chips Minieri had before the hand.

Before the cards are dealt, Minieri's expected number of chips he will have after the hand is just a.

After the cards are dealt, Minieri's expected number of chips is (a - 1600) + p x 3200, where p = 56.465%.

The 1600 is the amount of the big blind, and 3200 is the effective size of the pot, i.e. the size the pot would be if Minieri had just called.

So Minieri's equity gain due to luck = [a - 1600 + 56.465% x 3200] - a = -1600 + 1806.88 = 206.88.

Minieri's equity gain due to skill = the expected number of chips Minieri will have after the betting round is over minus the expected number he will have before the betting started.

After the betting round is over, Minieri has a + 4300.

Before the betting started, Minieri had a - 1600 chips plus equity in the pot worth p x 3200, so his expected number of chips before betting started was a - 1600 + p x 3200 = a + 206.88.

So, his equity gained due to skill = [a + 4300] - [a + 206.88] = 4093.12.

Hand 1. Lederer A♣ 7♠, Minieri 6♠ 6♦. Lederer 43.535%, Minieri 56.465%.

Lederer raises to 4300. Minieri raises to 47800. Lederer folds. Luck +206.88.   Skill +4093.12.

6. Another random walk example.

Suppose you either gain a chip or lose a chip each hand, each with prob. ½.

You have 3 chips. If you hit 0 chips you are eliminated. What is the probability that you last at least 7 more hands without being eliminated?

For this slide, let x mean "anything". We want y = P(go from (0,3) to (7,x) without hitting 0).

First, note that there is obviously some relationship between this and

P(go from (0,0) to (3,3) and then from (3,3) to (10,x) without hitting zero). In particular,

<u>P(go from (0,3) to (7,x) without hitting 0) = P(go from (3,3) to (10,x) not hitting zero).</u>

Consider a RW starting at (0,0). We know

P(go from (0,0) to (1,1) to (10,x) without hitting zero)

+ P(go from (0,0) to (1,-1) to (10,x) without hitting zero) = C(10,5) $½^{10}$.

And the two terms on the left are obviously equal, so this means we know

P(go from (0,0) to (1,1) to (10,x) without hitting zero) = C(10,5) $½^{11}$ = 12.30%.

Note that there are a couple ways this can happen.

12.30% = P(go from (0,0) to (1,1) to (2,2) to (3,3) to (10,x) without hitting zero)

+ P(go from (0,0) to (1,1) to (2,2) to (3,1) to (10,x) without hitting zero)

= ½ x ½ x ½ x P(go from (3,3) to (10,x) without hitting zero)

+ ½ x ½ x ½ x P(go from (3,1) to (10,x) without hitting zero)

= 1/8 y + 1/8 x P(go from (3,1) to (10,x) without hitting zero).

P(go from (2,0) to (3,1) to (10,x) without hitting zero) = ½ P(go from (2,0) to (10,x) without hitting zero)

= ½ P(go from (0,0) to (8,x) without hitting zero) = ½ P(go from (0,0) to (8,0)) = ½ C(8,4) $½^8$ = 13.67%.

So, 13.67% = P(go from (2,0) to (3,1) to (10,x) without hitting zero)

= ½ P(go from (3,1) to (10,x) without hitting zero).

So P(go from (3,1) to (10,x) without hitting zero) = 2 x 13.67% = 27.34%.

So we have 12.30% = 1/8 y + 1/8 (27.34%), and multiplying by 8, we get

y + 27.34% = 98.4%. y = 71.06%.

6. Another random walk example continued.

Suppose you either gain a chip or lose a chip each hand, each with prob. ½.

You have 3 chips. If you hit 0 chips you are eliminated. What is the probability that you last at least 7 more hands without being eliminated?

Do this in R.

```
f1 = function(){
    up1 = runif(7)<.5            ## up1 = 0s or 1s each with prob ½.
    up2 = up1*2-1                ## up2 are +1 or -1.
    c1 = 3+cumsum(up2)           ## c1 is your number of chips over time
    min(c1) > 0                  ## this is T or F, depending on whether you avoid 0 or not.
}
x = rep(TRUE,1000000)
for(i in 1:1000000) x[i] = f1()
x[1:1000]
mean(x)  ## 71.06%?
```