Stat 100a, Introduction to Probability.

Outline for the day:

1. Moment generating functions.
2. Review list.
3. Review problems.

Midterm 1 is one hour and 15 min, on Mon Jan31 in Haines A2.

Future classes will meet in Haines A2 unless otherwise noted.

Homework 2 is on the course website. It is due Mon Feb14 to STAT100AW22@stat.ucla.edu.

Your emails are on the course website, but I will take the emails off tonight at 9pm, so find the emails of your project teammates before 9pm.

http://www.stat.ucla.edu/~frederic/100A/W22

Moment generating functions, ch. 4.7

Suppose X is a random variable. $E(X)$, $E(X^2)$, $E(X^3)$, etc. are the *moments* of X.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Take derivatives with respect to t of $ø_X(t)$ and evaluate at t=0 to get moments of X.

1st derivative $(d/dt)\, e^{tX} = X\, e^{tX}$, $(d/dt)^2\, e^{tX} = X^2\, e^{tX}$, etc.

$(d/dt)^k\, E(e^{tX}) = E[(d/dt)^k\, e^{tX}] = E[X^k\, e^{tX}]$,   (see p.84)

so $ø'_X(0) = E[X^1\, e^{0X}] = E(X)$,

$ø''_X(0) = E[X^2\, e^{0X}] = E(X^2)$, etc.

The moment gen. function $ø_X(t)$ uniquely characterizes the distribution of X.

So to show that X is, say, Poisson, you just need to show that it has the moment generating function of a Poisson random variable.

Also, if X_i are random variables with cdfs F_i, and $ø_{X_i}(t) \to ø(t)$, where $ø_X(t)$ is the moment generating function of X which has cdf F, then $X_i \to X$ in distribution, i.e. $F_i(y) \to F(y)$ for all y where F(y) is continuous.

Moment generating functions, continued.

$ø_X(t) = E(e^{tX})$ is called the *moment generating function* of X.

Suppose X is 1 with probability 0.4 and 0 with probability 0.6. We call this Bernoulli (0.4). What is $ø_X(t)$? $E(e^{tX}) = (0.6)\,(e^{t(0)}) + (0.4)\,(e^{t(1)}) = 0.6 + 0.4\,e^t$.

Suppose X is Bernoulli (0.4) and Y is Bernoulli (0.7) and X and Y are independent.

What is the distribution of XY?

$ø_{XY}(t) = E(e^{tXY}) = P(XY=0)\,(e^{t(0)}) + P(XY=1)(e^{t(1)})$

$= P(X=0 \text{ or } Y=0)\,(1) + P(X=1 \text{ and } Y=1)e^t$

$= [1 - P(X=1)P(Y=1)] + P(X=1)P(Y=1)e^t$

$= [1 - 0.4 \text{ x } 0.7] + 0.4\text{x}0.7e^t$

$= 0.72 + 0.28e^t$, which is the moment generating function of a Bernoulli (0.28) random variable.

Therefore XY is Bernoulli (0.28).

What about Z = min{X,Y}?

If you think about it, Z = XY in this case, since X and Y are 0 or 1, so the answer is the same.

Review.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. $P(AB) = P(A)\,P(B|A)$ [$= P(A)P(B)$ if ind.]
7) Odds ratios.
8) Discrete RVs and probability mass function (pmf).
9) Expected value.
10) Pot odds calculations.
11) Luck and skill.
12) Variance and SD.
13) $E(aX+b)$, $E(X+Y)$, $V(X+Y)$.
14) Bayes's rule.
15) Markov and Chebyshev inequalities.
16) Moment generating functions.

We have basically done all of chapters 1-4.

on the midterm, FOR EACH QUESTION, WRITE THE LETTER OF YOUR ANSWER TO THE LEFT OF THE QUESTION.

Conditional probability.

When A and B are different outcomes on different collections of cards or different hands, then $P(B|A)$ can often be found directly.

But when A and B are outcomes on the same event, or same card, then sometimes it is helpful to use the definition $P(B|A) = P(AB)/P(A)$.

For example, let A = the event your hole cards are black, and let B = the event your hole cards are clubs.

$P(B|A) = P(AB)/P(A) = C(13,2)/C(52,2) / [C(26,2)/C(52,2)]$.

However, if A is the event your hole cards are black and B is the event the flop cards are all black, then $P(B|A) = C(24,3)/C(50,3)$ directly.

Example problems.

___ 1. What is the probability that you will be dealt a king and another card of the same suit as the king?

a. 1.69%. b. 3.62%. c. 4.89%. d. 5.02%. e. None of the above.

4 * 12 / C(52,2) = 3.62%.

P(flop an ace high flush)? [where the ace might be on the board]

-- 4 suits

-- one of the cards must be an ace. C(12,4) possibilities for the others.

So P(flop ace high flush) = 4 * C(12,4) / C(52,5)

= 0.0762%, or 1 in **1313**.

P(flop a straight | 87 of different suits in your hand)?

It could be 456, 569, 6910, or 910J. Each has 4*4*4 = 64 suit combinations.

So P(flop a straight | 87) = 64 * 4 / C(50,3)

= 1.31%.

P(flop a straight | 86 of different suits in your hand)?

Now it could be 457, 579, or 7910.

P(flop a straight | 86) = 64 * 3 / C(50,3)

= 0.980%.

Suppose X = 0 with probability ½, 1 with probability ¼, 2 with probability 1/8, and 3 with probability 1/8.
What is E(X)? What is $E(X^2)$? What is Var(X)? What is SD(X)? What is $ø_X(t)$?

$E(X) = 0(1/2) + 1(1/4) + 2(1/8) + 3(1/8) = 0.875.$

$E(X^2) = 0(1/2) + 1(1/4) + 4(1/8) + 9(1/8) = 1.875.$

$Var(X) = E(X^2) - \mu^2 = 1.875 - 0.875^2 = 1.11.$

$SD(X) = \sqrt{1.11} = 1.05.$

$ø_X(t) = E(e^{tX}) = ½ (1) + ¼ (e^t) + 1/8 (e^{2t}) + 1/8 (e^{3t}).$

P(flop two pairs).

If you're sure to be all-in next hand, what is P(you will flop two pairs)?

This is a tricky one. Don't double-count (4♠ 4♦ 9♠ 9♦ Q♦) and (9♠ 9♦ 4♠ 4♦ Q♦).

There are choose(13,2) possibilities for the NUMBERS of the two pairs.

For each such choice (such as 4 & 9),

there are choose(4,2) choices for the suits of the lower pair,

and the same for the suits of the higher pair.

So, choose(13,2) * choose(4,2) * choose(4,2) different possibilities for the two pairs.

For each such choice, there are 44 [52 - 8 = 44] different possibilities for your fifth card, so that it's not a full house but simply two pairs. So,

P(flop two pairs) = C(13,2) * C(4,2) * C(4,2) * 44 / C(52,5)

~ 4.75%, or 1 in **21**.

P(flop two pairs).

Here is another way to do it. Find the mistake.

P(flop 2 pairs) = P(pocket pair and flop 2 pairs) + P(no pocket pair and flop 2 pairs)

= P(pocket pair) P(flop 2 pairs | pocket pair) + P(no pocket pair) P(flop 2 pairs | no pocket pair)

= P(pocket pair aa) * P(bbc | aa) + P(ab)*P(abc | ab)

= 13*C(4,2)/C(52,2) * 12*C(4,2)*44/C(50,3) + C(13,2)*4*4/C(52,2) * 3*3*44/C(50,3)

= 2.85%.

What is the problem here?

P(flop two pairs).

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= P(pocket pair aa) * P(bbc | aa) + P(ab)*P(abc | ab)

= 13*C(4,2)/C(52,2) * 12*C(4,2)*44/C(50,3) + C(13,2)*4*4/C(52,2) * **3*3*44**/C(50,3)

= 2.85%.

What is the problem here?

P(flop 2 pairs | no pocket pair) ≠ P(ab)*P(abc | ab). If you have ab, it could come acc or bcc on the flop.

13*C(4,2)/C(52,2) * 12*C(4,2)*44/C(50,3) + C(13,2)*4*4/C(52,2) * **(3*3*44 + 6*11*C(4,2))** /C(50,3)

= 4.75%.

A = the event you have AK.

B = the event both your hole cards are clubs.

What is P(B|A)? Are A and B independent?

P(B|A) = P(A and B) / P(A)

= P(A♣ K♣) / P(AK)

= [1/C(52,2)] / [16/C(52,2)]

= 1/16.

P(B) = P(♣ ♣) = C(13,2)/C(52,2) = 1/17.

Not independent.