Stat 100a, Introduction to Probability.

Outline for the day.

- 1. Conditional expectation.
- 2. Law of large numbers.
- 3. Example problems.
- 4. Review list.

Bring a PEN or PENCIL and CALCULATOR and any books or notes you want to the exams.

Conditional expectation, E(Y | X), ch. 7.2. Suppose X and Y are discrete. Then E(Y | X=j) is defined as $\sum_{k} k P(Y = k | X = j)$, just as you'd think. E(Y | X) is a **random variable** such that E(Y | X) = E(Y | X=j) whenever X = j.

For example, let X = the # of spades in your hand, and Y = the # of clubs in your hand. **a**) What's E(Y)? **b**) What's E(Y|X)? **c**) What's P(E(Y|X) = 1/3)?

a.
$$E(Y) = 0P(Y=0) + 1P(Y=1) + 2P(Y=2)$$

= 0 + $13x39/C(52,2) + 2C(13,2)/C(52,2) = 0.5$

b. X is either 0, 1, or 2. If X = 0, then E(Y|X) = E(Y | X=0) and E(Y | X=0) = 0 P(Y=0 | X=0) + 1 P(Y=1 | X=0) + 2 P(Y=2 | X = 0) = 0 + 13x26/C(39,2) + 2 C(13,2) / C(39,2) = 2/3. E(Y | X=1) = 0 P(Y=0 | X=1) + 1 P(Y=1 | X=1) + 2 P(Y=2 | X = 1) = 0 + 13/39 + 2 (0) = 1/3. E(Y | X=2) = 0 P(Y=0 | X=2) + 1 P(Y=1 | X=2) + 2 P(Y=2 | X = 2) = 0 + 1 (0) + 2(0) = 0.So E(Y | X = 0) = 2/3, E(Y | X = 1) = 1/3, and E(Y | X = 2) = 0. That's what E(Y|X) is c. P(E(Y|X) = 1/3) is just $P(X=1) = 13x39/C(52,2) \sim 38.24\%.$ Law of Large Numbers (LLN) and the Fundamental Theorem of Poker, ch 7.3. David Sklansky, *The Theory of Poker*, 1987.

"Every time you play a hand differently from the way you would have played it if you could see all your opponents' cards, they gain; and every time you play your hand the same way you would have played it if you could see all their cards, they lose. Conversely, every time opponents play their hands differently from the way they would have if they could see all your cards, you gain; and every time they play their hands the same way they would have played if they could see all your cards, you lose."

Meaning?

LLN: If X_1, X_2 , etc. are iid with expected value μ and sd σ , then $X_n \longrightarrow \mu$. Any short term good or bad luck will ultimately become *negligible* to the sample mean. However, this does not mean that good luck and bad luck will ultimately cancel out. See p132.

Review list.

- 1) Basic principles of counting.
- 2) Axioms of probability, and addition rule.
- 3) Permutations & combinations.
- 4) Conditional probability.
- 5) Independence.
- 6) Multiplication rules. P(AB) = P(A) P(B|A) [= P(A)P(B) if ind.]
- 7) Odds ratios.
- 8) Discrete RVs and probability mass function (pmf).
- 9) Expected value.
- 10) Pot odds calculations.
- 11) Luck and skill.
- 12) Variance and SD.
- 13) E(aX+b) and E(X+Y).
- 14) Bayes's rule.
- 15) Markov and Chebyshev inequalities.
- 16) Bernoulli, binomial, geometric, Poisson, and negative binomial rvs.
- 17) Moment generating functions.
- 18) pdf, cdf, uniform, exponential, normal and Pareto rvs. F'(y) = f(y).
- 19) Survivor functions.
- 20) Covariance and correlation.
- 21) Bivariate normal.
- 22) Conditional expectation.
- 23) Law of large numbers.

We have basically done all of chapters 1-7.3. Ignore 6.7 and most of 6.3 on optimal play.

Example problems.

X is a continuous random variable with cdf $F(y) = 1 - y^{-1}$, for y in $(1,\infty)$, and F(y) = 0 otherwise. a. What is the pdf of X? b. What is f(1)? c. What is E(X)?

a. $f(y) = F'(y) = d/dy (1 - y^{-1}) = y^{-2}$, for y in $(1,\infty)$, and f(y) = 0 otherwise.

To be a pdf, f(y) must be nonnegative for all y and integrate to 1. $f(y) \ge 0$ for all y, and $\int_{-\infty}^{\infty} f(y)dy = \int_{1}^{\infty} y^{-2} dy = -y^{-1}]_{1}^{\infty} = 0 + 1 = 1$. So, f is indeed a pdf.

b. $f(1) = 1^{-2} = 1$.

c. E(X) = $\int_{-\infty}^{\infty} y f(y) dy = \int_{1}^{\infty} y y^{-2} dy = \int_{1}^{\infty} y^{-1} dy = \ln(\infty) - \ln(1) = \infty$. This can happen.

X is a continuous random variable with cdf $F(y) = 1 - y^{-2}$, for y in $(1,\infty)$, and F(y) = 0 otherwise. a. What is the pdf of X? b. What is f(1)? Is this a problem? c. What is E(X)? d. What is $P(2 \le X \le 3)$? e. What is P(2 < X < 3)?

a.
$$f(y) = F'(y) = d/dy (1 - y^{-2}) = 2y^{-3}$$
, for y in $(1,\infty)$, and $f(y) = 0$ otherwise.

To be a pdf, f(y) must be nonnegative for all y and integrate to 1. $f(y) \ge 0$ for all y, and $\int_{-\infty}^{\infty} f(y)dy = \int_{1}^{\infty} 2y^{-3} dy = -y^{-2}]_{1}^{\infty} = 0 + 1 = 1$. So, f is indeed a pdf.

b. f(1) = 2. This does not mean P(X=1) is 2. It is not a problem.

c. E(X) =
$$\int_{-\infty}^{\infty} y f(y) dy = 2 \int_{1}^{\infty} y y^{-3} dy = 2 \int_{1}^{\infty} y^{-2} dy = -2y^{-1} \Big]_{1}^{\infty} = 0 + 2 = 2.$$

d. P(2 ≤ X ≤ 3) =
$$\int_2^3 f(y) dy = 2 \int_2^3 y^{-3} dy = -y^{-2} \Big]_2^3 = -1/9 + 1/4 \sim 0.139.$$

Alternatively, $P(2 \le X \le 3) = F(3) - F(2) = 1 - 1/9 - 1 + 1/4 \sim 0.139$. e. Same thing. Suppose X is uniform(0,1), Y is exponential with E(Y)=2, and X and Y are independent. What is cov(3X+Y, 4X-Y)?

cov(3X+Y, 4X-Y) = 12 cov(X,X) - 3cov(X,Y) + 4cov(Y,X) - cov(Y,Y)= 12 var(X) - 0 + 0 - var(Y).

For exponential, $E(Y) = 1/\lambda$ and $var(Y) = 1/\lambda^2$, so $\lambda = 1/2$ and var(Y) = 4. What about var(X)? $E(X^2) = \int y^2 f(y) dy$ $= \int_0^1 y^2 dy$ because f(y) = 1 for uniform(0,1) for y in (0,1), $= y^3/3]_0^1$ = 1/3. $var(X) = E(X^2) - \mu^2 = 1/3 - \frac{1}{4} = 1/12$.

$$cov(3X+Y, 4X-Y) = 12 (1/12) - 4$$

= -3.

What is the moment generating function of a uniform random variable?

If X is uniform(a,b), then it has density f(x) = 1/(b-a) between a and b, and f(x) = 0 for all other x. $\emptyset_X(t) = E(e^{tX})$ $= \int_a^b e^{tx} f(x) dx$ $= \int_a^b e^{tx} 1/(b-a) dx$ $= 1/(b-a) \int_a^b e^{tx} dx$ $= 1/(b-a) e^{tx}/t]_a^b dx$ $= (e^{tb} - e^{ta})/[t(b-a)].$