Stat 100a: Introduction to Probability.

Outline for the day:

1. Hand in hw2.
2. Midterm.
3. Markov and Chebyshev inequalities.
4. "Unbeatable" Texas Holdem strategy.
5. Odds ratios vs. probabilities, Gold vs. Hellmuth.
6. Sums of random variables.
7. Farha vs. Antonius, expected value and variance.
8. Rainbow flops.
9. Hw3 problems.

Read through chapter 7.1 this week!

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1. Hand in hw2.
2. Midterm.

The midterm is Tue Aug 26, in class, from 10-1130.
It will be on chapters 1-6 and 7.1.
I may lecture on some stuff in chapter 7.2 before then, but the midterm will only cover ch 1-7.1.
It will be mostly multiple choice, plus a short answer question.
You can use the book, plus just one page (double sided) of 8.5 x 11 paper with notes on it. Keep that page of notes after the test.
Next lecture we will do mostly review and practice problems.

I do not advise guessing none of the above!
For additional practice problems, see Sheldon Ross's book "A First Course in Probability."

OH today 1145-1215, next week 1130-12.

Statistics 100a Midterm

Rick Paik Schoenberg, 8/26/14, 10am-11:30am.

PRINT YOUR NAME:

SIGN YOUR NAME:

Do not turn the page and start the exam until you are told to do so.

There are 16 multiple choice questions worth 6 points each and 1 short-answer question worth 4 points.

Final answers are rounded to 3 significant digits.

No partial credit is given for multiple choice questions. Circle one answer only. If your answer is none of the above, then indicate the correct answer.

For problems 4-11, assume that you are guaranteed to be all-in next hand, no matter what. Also, assume that all of these questions relate specifically to the *next hand only*. In other words, when I ask what the probability is that you will be dealt a certain type of hand, I don't mean *eventually*; I mean on the very next hand.

7s means the seven of spades. Qh means the queen of hearts, etc.

3. Markov and Chebyshev inequalities. Ch 4.6.

The Markov inequality states

If X takes only non-negative values, and c is any number > 0, then

$P(X \geq c) \leq E(X)/c$.

Proof. The discrete case is given on p82.

Here is a proof for the case where X is continuous with pdf f(y).

$E(X) = \int y\, f(y)\, dy$

$= \int_0^c yf(y)dy + \int_c^\infty yf(y)dy$

$\geq \int_c^\infty yf(y)dy$

$\geq \int_c^\infty cf(y)dy$

$= c \int_c^\infty f(y)dy$

$= c\, P(X \geq c)$.

Thus, $P(X \geq c) \leq E(X) / c$.

The Chebyshev inequality states

For any random variable Y with expected value μ and variance σ^2, and any real number a > 0,

$P(| Y - \mu | \geq a) \leq \sigma^2 / a^2$.

Proof. The Chebyshev inequality follows directly from the Markov equality by letting $c = a^2$ and $X = (Y-\mu)^2$.

Examples of the use of the Markov and Chebyshev inequalities are on p83.

4. "Unbeatable Texas Holdem Strategy"

http://www.freepokerstrategy.com : all in with AK-AT or any pair.

P(getting such a hand) = 4 x [16/1326] + 13 x [6/1326]

= 4 x 1.2% + 13 x 0.45% = 10.7%.

Say you're dealt 100 hands. Pay ~11 blinds = $55.

Expect 10.7 (~ 11) such good hands.

Say you're called by 88-AA, and AK, for $100 on avg.

P(player 1 has one of these) = 7 x 0.45% + 1.2% = 4.4%.

P(of 8 opponents, someone has one of these) ~ 1 - $(95.6\%)^8$ = 30%.

So, you win pre-flop 70% of the time. (Say $10 on avg.)

= 11 x 70% x $10 = $77 profit.

Other 30%, you're on avg about a 65-35 underdog, so you

win 11 x 30% x 35% x $100 = $115.50

lose 11 x 30% x 65% x $100 = $214.50.

Total: exp. to win $77 + $115.50 - $55 - $214.50 = **-$77**/ 100 hands.

More example code for project A

```
unbeatable1 = function(numattable1, crds1, board1,  round1,
currentbet, mychips1, pot1, roundbets, blinds1, chips1, ind1,
dealer1, tablesleft){
    ## any pair, or AT-AK,
    a1 = 0
    if((crds1[1,1] == crds1[2,1]) || ((crds1[1,1] > 13.5) &&
            (crds1[2,1]>9.5))) a1 = mychips1
    a1
} ## end of unbeatable1
```

Winning code last time.

```
legendary = function(numattable1, crds1, board1,  round1, currentbet, mychips1,
                                   pot1, roundbets, blinds1, chips1, ind1, dealer1, tablesleft){
    ## all in if pair of 9s or better,
    ## or same suit with the lower card is 10 or above,
    ## or different suit with lower card is Jack or above,
    ## or you have less than 4 times big blind and at least one card is Jack or higher.
    a1 = 0
    if((crds1[1,1] == crds1[2,1]) && (crds1[2,1] > 8.5)) a1 = mychips1
    if((crds1[1,2] == crds1[2,2]) && (crds1[2,1] > 9.5)) a1 = mychips1
    if((crds1[2,1] > 10.5)) a1 = mychips1
    if((mychips1 < 4 * blinds1) && (crds1[1,1] > 10.5))  a1 = mychips1
    a1
} ## end of legendary
```

```
zebra = function(numattable1, crds1, board1,  round1, currentbet, mychips1, pot1,roundbets, blinds1, chips1,
ind1, dealer1, tablesleft){
   ## if pair of 10s or higher, all in for sure, no matter what.  If AK or AQ, all in with probability 75%.
   ## if pair of 7s or higher and there are 6 or fewer players at your table (including you), then all in.
   ## if your chip count is less than twice the big blind, go all in with any cards.
   ## if nobody's raised yet ... and there are 3 or fewer players left behind you, go all in with any pair or ace.
   ##                               ... and there's only 1 or 2 players behind you, then go all in with any cards.
   a1 = 0
   x = runif(1)            ## x is a random number between 0 and 1.
   y = max(roundbets[,1])     ## y is the maximum bet so far.
   big1 = dealer1 + 2
   if(big1 > numattable1) big1 = big1 - numattable1
   z = big1 - ind1
   if(z<0) z = z + numattable1
   ## the previous 4 lines make it so z is the number of players left to act behind you.
   if((crds1[1,1] == crds1[2,1]) && (crds1[2,1] > 9.5)) a1 = mychips1
   if((crds1[1,1] == 14) && (crds1[1,2]>11.5) && (x<.75)) a1 = mychips1
   if((crds1[1,1] == crds1[2,1]) && (crds1[2,1] > 6.5) && (numattable1 < 6.5)) a1 = mychips1
   if(mychips < 2*blinds1) a1 = mychips1
   if(y <= blinds1){  if((z < 3.5) && ((crds1[1,1] == crds1[2,1]) || (crds1[1,1] == 14))) a1 = mychips1
              if(z < 2.5) a1 = mychips1
              }
   a1} ## end of zebra
```

5. Odds ratios revisited:

Odds ratio of A = $P(A)/P(A^c)$

Odds *against* A = Odds ratio of A^c = $P(A^c)/P(A)$.

An advantage of probability over odds ratios is the multiplication rule:

$P(A \& B) = P(A) \times P(B|A)$, but you can't multiply odds ratios.

Example: Gold vs. Hellmuth on High Stakes Poker….

Gold: A♣ K♥.   Hellmuth: A♠ K♠. Farha: 8♥ 7♣.   Flop: 4♠ 7♠ K♣.

Given these 3 hands and the flop, what is P(Hellmuth makes a flush)?

43 cards left: 9 ♠s, 34 non-♠s. Of choose(43, 2) = 903 eq. likely turn/river combos,

choose(9,2) = 36 have both ♠s, and 9 *34 = 306 have exactly 1 ♠. 342/903 = 37.9%.

So, P(Hellmuth *fails* to make a flush) = 100% - 37.9% = 62.1%.

Gold: A♣ K♥.   Hellmuth: A♠ K♠. Farha: 8♥ 7♣.   Flop: 4♠ 7♠ K♣.

P(Hellmuth *fails* to make a flush) = 100% - 37.9% = 62.1%.

Alt.: Given these 3 hands and the flop, P(neither turn nor river is a ♠)

= P(turn is non-♠ AND river is non-♠)

= P(turn is non-♠) * P(river is non-♠ | turn is non-♠)   [P(A&B) = P(A)P(B|A)]

= 34/43 * 33/42 = 62.1%.

Note that we can multiply these probabilities: 34/43 * 33/42 = 62.1%.

What are the *odds against* Hellmuth failing to make a flush?

37.9% ÷ 62.1% = 0.61 : 1.

Odds against non-♠ on turn = 0.26 :1.

Odds against non-♠ on river | non-♠ on turn : 0.27 : 1.

0.26 * 0.27 = 0.07. Nowhere near the right answer.

Moral: you can't multiply odds ratios!

6. E(X+Y) = E(X) + E(Y). pp126-127.

A fact given in ch.7 is that E(X+Y) = E(X) + E(Y), for any random variables X and Y, *whether X & Y are independent or not*, as long as E(X) and E(Y) are finite.

Similarly, E(X + Y + Z + …) = E(X) + E(Y) + E(Z) + …

And, if X & Y are independent, then V(X+Y) = V(X) + V(Y).

so SD(X+Y) = √[SD(X)^2 + SD(Y)^2].

<u>Example 1</u>: Play 10 hands. X = your total number of pocket aces. What is E(X)?

X is binomial (n,p) where n=10 and p = 0.00452, so E(X) = np = 0.0452.

Alternatively, X = # of pocket aces on hand 1 + # of pocket aces on hand 2 + …

So, E(X) = Expected # of AA on hand1 + Expected # of AA on hand2 + …

Each term on the right = 1 * 0.00452 + 0 * 0.99548 = 0.00452.

So E(X) = 0.00452 + 0.00452 + … + 0.00452 = 0.0452.

<u>Example 2</u>: Play 1 hand, against 9 opponents. X = total # of pocket aces at the table. E (X) = ?

Note: not independent! If you have AA, then it's unlikely anyone else does too.

Nevertheless, Let X_1 = 1 if player #1 has AA, and 0 otherwise.

X_2 = 1 if player #2 has AA, and 0 otherwise, and so on.

Then $X = X_1 + X_2 + \ldots + X_{10}$. So $E(X) = E(X_1) + E(X_2) + \ldots + E(X_{10})$

= 0.00452 + 0.00452 + … + 0.00452 = 0.0452.

7. Farha vs. Antonius, expected value and variance.

$E(X+Y) = E(X) + E(Y)$. *Whether X & Y are independent or not*!
Similarly, $E(X + Y + Z + \ldots) = E(X) + E(Y) + E(Z) + \ldots$
And, if X & Y are independent, then $V(X+Y) = V(X) + V(Y)$.
so $SD(X+Y) = \sqrt{[SD(X)^2 + SD(Y)^2]}$.
Also, if $Y = 9X$, then $E(Y) = 9E(Y)$, and $SD(Y) = 9SD(X)$. $V(Y) = 81V(X)$.

Farha vs. Antonius.

Running it 4 times. Let X = chips you have after the hand. Let p be the prob. you win.
$X = X_1 + X_2 + X_3 + X_4$, where X_1 = chips won from the first "run", etc.
$E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$
= 1/4 pot (p) + 1/4 pot (p) + 1/4 pot (p) + 1/4 pot (p)
= pot (p)
= same as E(Y), where Y = chips you have after the hand if you ran it once!

But the SD is smaller: clearly $X_1 = Y/4$, so $SD(X_1) = SD(Y)/4$. So, $V(X_1) = V(Y)/16$.
$V(X) \sim V(X_1) + V(X_2) + V(X_3) + V(X_4)$,
$= 4\ V(X_1)$
$= 4\ V(Y) / 16$
$= V(Y) / 4$. **So SD(X) = SD(Y) / 2.**

8. Rainbow flops.

P(Rainbow flop) = choose(4,3) * 13 * 13 * 13 ÷ choose(52,3)

choices for the 3 suits — numbers on the 3 cards — possible flops

~ 39.76%.

Q: Out of 100 hands, what is the expected number of rainbow flops? +/- what?

X = Binomial (n,p), with n = 100, p = 39.76%, q = 60.24%.

E(X) = np = 100 * 0.3976 = 39.76

SD(X) = $\sqrt{(npq)}$ = sqrt(23.95) = 4.89.

So, expect around 39.76 +/- 4.89 rainbow flops, out of 100 hands.

Rainbow flops, continued.

P(Rainbow flop) ~ 39.76%.

Q: Let X = the number of hands til your 4th rainbow flop.

What is P(X = 10)? What is E(X)? What is SD(X)?

X = negative binomial (r,p), with r = 4, p = 39.76%, q = 60.24%.

$P(X = k) = \text{choose}(k-1, r-1)\ p^r\ q^{k-r}$.

Here k = 10. $P(X = 10) = \text{choose}(9,3)\ 39.76\%^4\ 60.24\%^6 = 10.03\%$.

$\mu = E(X) = r/p = 4 \div 0.3976 = 10.06$ hands.

$\sigma = SD(X) = (\sqrt{rq}) / p =$ sqrt(4*0.6024) / 0.3976 = 3.90 hands.

So, you expect it *typically* to take around

10.06 +/- 3.90 hands til your 4th rainbow flop.

9. Hw3 problems.

6.12 and 7.14b are hard.

For 6.12, there are different ways to do this problem, but one way leads you to some tough integrals. You can use the fact that $\int c\, [2\exp(-2c)]\, dc$ is the expected value of an exponential random variable with parameter $\lambda = 2$, and this expected value is 1/2. Similarly, $\int c \exp(-c)\, dc$ is the expected value of an exponential with $\lambda = 1$, which is 1.

For 7.14b, let $x = r^2$, and you'll get an equation like $-x^3 + 2x - 1 = 0$, so $(x-1)(-x^2 - x + 1) = 0$. There are 3 possible solutions to this: $x = 1$, or $x^2 + x - 1 = 0$, so $x = [-1 \pm \sqrt{(1 + 4)}]/2$. Now reason why we can rule out 2 of these possibilities for x.

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