Stat 100a: Introduction to Probability.

<u>Outline for the day:</u>

1. Gold vs. Farha and Bayes's rule.
2. Luck and skill.
3. Yang and Kravchenko.
4. P(flop an ace high flush), P(flop a straight flush), etc.
5. Review list.
6. Cards til 2^{nd} king.
7. P(2 pairs).
8. P(suited King).
9. Hellmuth vs. Farha.
10. Covariance and correlation.
11. Pareto random variables.
12. Deal making.

Read through chapter 7.1.

♣ ♥ ♦ ♠

1. Gold vs. Farha. Gold: 10♦ 7♠ Farha: Q♦ Q♠ Flop: 9♦ 8♥ 7♣

a) Who really is the favorite (ignoring all other players' cards)?

Gold's outs: J, 6, 10, 7. (4 + 4 + 3 + 2 = 13 outs, 32 non-outs)

P(Gold wins) = P(Out Out or Jx [$x \neq 10$] or 6x or 10y [$y \neq Q,9,8$] or 7z [$z \neq Q$])

= [choose(13,2) + 4*28 + 4*32 + 3*24 + 2*30] ÷ choose(45,2)

= 450 ÷ 990 = 45.45%.

b) What would you guess Gold had?

Say he'd do that 50% of the time with a draw,

100% of the time with an overpair,

and 90% of the time with two pairs. (and that's it)

Using Bayes' rule, P(Gold has a DRAW | Gold raises ALL-IN)

$$= \frac{[P(\text{all-in} \mid \text{draw}) * P(\text{draw})]}{[P(\text{all-in} \mid \text{draw})\, P(\text{draw})] + [P(\text{all-in} \mid \text{overpair})\, P(\text{overpair})] + [P(\text{all-in} \mid \text{2pairs})\, P(\text{2 pairs})]}$$

= [50% * P(draw)] ÷ [50% * P(draw)] + [100% * P(overpair)] + [90% * P(2 pairs)]

2. Luck vs. skill. pp 71-79. Any thoughts?

3. Yang / Kravchenko.

Yang A♠ 10♦. Pot is 19million. Bet is 8.55 million.

Needs P(win) > 8.55 ÷ (8.55 + 19) = 31%.

vs. AA: 8.5%. AJ-AK: 25-27%. KK-TT: 29%. 99-22: 44-48%. KQs: 56%.

Bayesian method: average these probabilities, weighting each by its *likelihood*.

See p49-53.

Yang / Kravchenko.

Yang A♠ 10♦. Pot is 19.0 million. Bet is 8.55 million.

Suppose that, averaging the different probabilities, P(Yang wins) = 30%.

And say Yang *calls*. Let X = the number of chips Kravchenko has after the hand.

What is E(X)? [Note, if Yang *folds*, then X = **19.0** million for sure.]

E(X) $= \sum$ [k * P(X=k)]

= [0 * 30%] + [27.55 million * 70%]

= **19.285** million.

4. P(flop an ace high flush)? [where the ace might be on the board]

-- 4 suits

-- one of the cards must be an ace. choose(12,4) possibilities for the others.

So P(flop ace high flush) = 4 * choose(12,4) / choose(52,5)

= 0.0762%, or 1 in **1313**.

P(flop a straight flush)?

-- 4 suits

-- 10 different straight-flushes in each suit. (5 high, 6 high, …, Ace high)

So P(flop straight flush) = 4 * 10 / choose(52,5)

= 0.00154%, or 1 in **64974**.

5. <u>Review</u> List. We have basically done up through ch7.1.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. $P(AB) = P(A)\,P(B|A)$ [$= P(A)P(B)$ if ind.]
7) Odds ratios.
8) Random variables (RVs).
9) Discrete and continuous RVs, pmf, cdf, pdf, survivor function.
10) Expected value, variance, and SD.
11) Pot odds calculations.
12) Bayes's rule.
13) Bernoulli RV. [0-1. $\mu = p$, $\sigma = \sqrt{(pq)}$.]
14) Binomial RV. [# of successes, out of n tries. $\mu = np$, $\sigma = \sqrt{(npq)}$.]
15) Geometric RV. [# of tries til 1st success. $\mu = 1/p$, $\sigma = (\sqrt{q}) / p$.]
16) Negative binomial RV. [# of tries til rth success. $\mu = r/p$, $\sigma = (\sqrt{rq}) / p$.]
17) Poisson RV. [$\mu = \lambda$, $\sigma = \sqrt{\lambda}$.]
18) $E(X+Y)$, $V(X+Y)$, and $E[f(X)\,g(Y)]$ when X and Y are ind.
19) Moment generating functions.
20) Uniform, exponential, and normal random variables.
21) Markov and Chebyshev inequalities.

6. Cards til 2nd king.

Deal the cards face up, without reshuffling.

Let Z = the number of cards til the 2nd king.

What is E(Z)?

The solution uses the fact

$$E(X+Y+Z + \ldots) = E(X) + E(Y) + E(Z) + \ldots.$$

E(cards til 2^{nd} king).

Z = the number of cards til the 2nd king. What is E(Z)?

Let X_1 = number of non-king cards before 1^{st} king.

Let X_2 = number of non-kings after 1^{st} king til 2^{nd} king.

Let X_3 = number of non-kings after 2^{nd} king til 3^{rd} king.

Let X_4 = number of non-kings after 3^{rd} king til 4^{th} king.

Let X_5 = number of non-kings after 4^{th} king til the end of the deck.

Clearly, $X_1 + X_2 + X_3 + X_4 + X_5 + 4 = 52$.

By symmetry, $E(X_1) = E(X_2) = E(X_3) = E(X_4) = E(X_5)$.

Therefore, $E(X_1) = E(X_2) = 48/5$.

$Z = X_1+X_2+2$, so $E(Z) = E(X_1) + E(X_2) + 2 = 48/5 + 48/5 + 2 = 21.2$.

7. P(flop two pairs).

If you're sure to be all-in next hand, what is P(you will flop two pairs)?

This is a tricky one. Don't double-count (4♣ 4♥ 9♣ 9♥ Q♣) and (9♣ 9♥ 4♣ 4♥ Q♣)!

There are choose(13,2) possibilities for the NUMBERS of the two pairs.

For each such choice (such as 4 & 9),

there are choose(4,2) choices for the suits of the lower pair,

and same for the suits of the higher pair.

So, choose(13,2) * choose(4,2) * choose(4,2) different possibilities for the two pairs.

For each such choice, there are 44 [52 - 8 = 44] different possibilities for your fifth card, so that it's not a full house but simply two pairs. So,

P(flop two pairs) = choose(13,2) * choose(4,2) * choose(4,2) * 44 / choose(52,5)

~ 4.75%, or 1 in **21**.

8. What is the probability that you will be dealt a king and another card of the same suit as the king?

4 * 12 / C(52,2) = 3.62%.

The typical number of hands until this occurs is ...

1/.0362 ~ 27.6.
(√96.38%) / 3.62% ~ 27.1.
So the answer is 27.6 +/- 27.1.

9. Hellmuth vs. Farha.

P(Hellmuth makes a flush)

$$= \frac{C(11,5) + C(11,4) * 37 + C(11,3) * C(37,2)}{C(48,5)} = 7.16\%.$$

P(Farha makes a flush)

$$= \frac{2 * (C(12,5) + C(12,4) * 36)}{C(48,5)} = 2.17\%.$$

10. Covariance and correlation, p127.

For any random variables X and Y,

$\text{var}(X+Y) = E[(X+Y)]^2 - [E(X) + E(Y)]^2$

$\quad = E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2 + 2E(XY) - 2E(X)E(Y)$

$\quad = \text{var}(X) + \text{var}(Y) + 2[E(XY) - E(X)E(Y)].$

$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$ is called the *covariance* between X and Y,

$\text{cor}(X,Y) = \text{cov}(X,Y) / [SD(X)\ SD(Y)]$ is called the *correlation* bet. X and Y.

If X and Y are ind., then $E(XY) = E(X)E(Y)$,

so $\text{cov}(X,Y) = 0$, and $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$.

Just as $E(aX + b) = aE(X) + b$, for any real numbers a and b,

$\text{cov}(aX + b,Y) = E[(aX+b)Y] - E(aX+b)E(Y)$

$= aE(XY) + bE(Y) - [aE(X)E(Y) + bE(Y)] = a\ \text{cov}(X,Y)$.

Ex. 7.1.3 is worth reading.

X = the # of 1^{st} card, and Y = X if 2^{nd} is red, -X if black.

$E(X)E(Y) = (8)(0)$.

$P(X = 2 \text{ and } Y = 2) = 1/13 * ½ = 1/26$, for instance, and same with any other combination,

so $E(XY) = 1/26\ [(2)(2)+(2)(-2)+(3)(3)+(3)(-3) + ... + (14)(14) + (14)(-14)]$
$= 0$.

So X and Y are *uncorrelated*, i.e. $\text{cor}(X,Y) = 0$.

But X and Y are not independent.

$P(X=2 \text{ and } Y=14) = 0$, but $P(X=2)P(Y=14) = (1/13)(1/26)$.

11. Pareto random variables. ch6.6

Pareto random variables are an example of *heavy-tailed* random variables, which means they have very, very large outliers much more frequently than other distributions like the normal or exponential.

For a Pareto random variable, the pdf is $f(y) = (b/a)\,(a/y)^{b+1}$, and the cdf is

$F(y) = 1 - (a/y)^b$,

for $y>a$, where $a>0$ is the *lower truncation point*, and $b>0$ is a parameter called the *fractal dimension*.

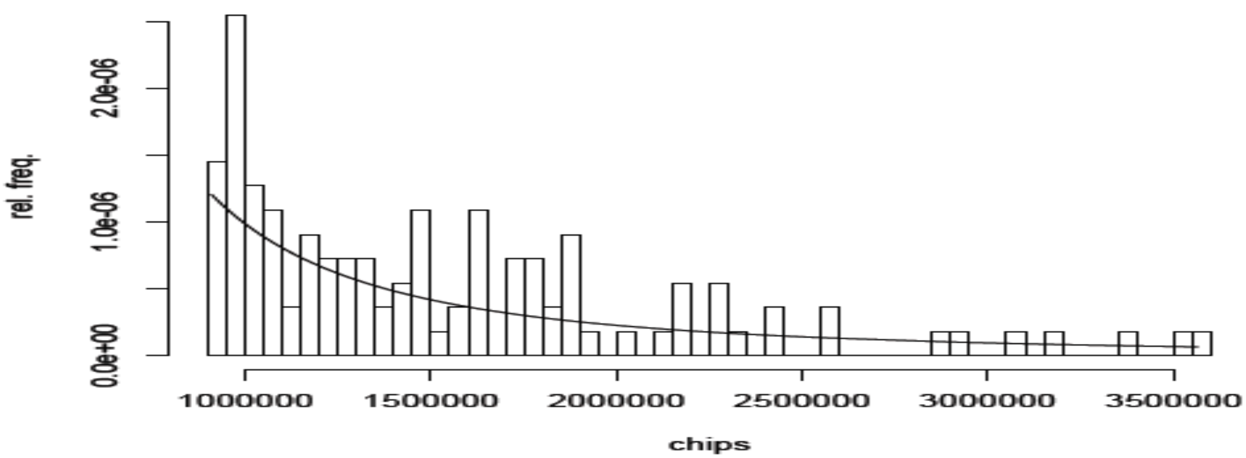

Figure 6.6.1: Relative frequency histogram of the chip counts of the leading 110 players in the 2010 WSOP main event after day 5. The curve is the Pareto density with $a = 900{,}000$ and $b = 1.11$.

12. Deal-making. (Expected value, game theory)

Game-theory: For a symmetric-game tournament, the probability of winning is approx. optimized by the *myopic rule* (in each hand, maximize your expected number of chips),

and

P(you win) = your proportion of chips (Theorems 7.6.6 and 7.6.7 on pp 151-152).

For a *fair* deal, the amount you win = the *expected value* of the amount you will win. See p61.

For instance, suppose a tournament is winner-take-all, for \$8600.

With 6 players left, you have 1/4 of the chips left.

An *EVEN SPLIT* would give you \$8600 ÷ 6 = **\$1433**.

A *PROPORTIONAL SPLIT* would give you \$8600 x (your fraction of chips)

= \$8600 x (1/4) = \$**2150**.

A *FAIR DEAL* would give you the *expected value* of the amount you will win

= \$8600 x P(you get 1st place) = \$**2150**.

But suppose the tournament is not winner-take-all, but pays

\$3800 for 1st, \$2000 for 2nd, \$1200 for 3rd, \$700 for 4th, \$500 for 5th, \$400 for 6th.

Then a *FAIR DEAL* would give you

\$3800 x P(1st place) + \$2000 x P(2nd) +\$1200 x P(3rd)+\$700xP(4th) +\$500xP(5th) +\$400xP(6th).

Hard to determine these probabilities. But, P(1st) = 25%, and you might roughly estimate the others as P(2nd) ~ 20%, P(3rd) ~ 20%, P(4th) ~ 15%, P(5th) ~10%, P(6th) ~ 10%, and get

\$3800 x 25% + \$2000 x 25% +\$1200 x 20% + \$700x 15% + \$500x 10% +\$400x 5% = **\$1865**.

If you have 40% of the chips in play, then:

EVEN SPLIT = **\$1433**.

PROPORTIONAL SPLIT = **\$3440**.

FAIR DEAL ~ **\$2500**!

Another example. Before the Wasicka/Binger/Gold hand,

Gold had 60M, Wasicka 18M, Binger 11M.

Payouts: 1st place $12M, 2nd place $6.1M, 3rd place $4.1M.

Proportional split: of the total prize pool left, you get your proportion of chips in play.

e.g. $22.2M left, so Gold gets 60M/(60M+18M+11M) x $22.2M ~ $15.0M.

A *fair* deal would give you

P(you get 1st place) x $12M + P(you get 2nd place) x $6.1M + P(3rd pl.) x $4.1M .

*Even split: Gold $7.4M, Wasicka $7.4M, Binger $7.4M.

*Proportional split: Gold **$15.0M**, Wasicka $4.5M, Binger **$2.7M**.

*Fair split: Gold $10M, Wasicka $6.5M, Binger $5.7M.

*End result: Gold $12M, Wasicka $6.1M, Binger $4.1M.