Stat 100a: Introduction to Probability.

Outline for the day:

1. Review list.
2. Random walk example.
3. Bayes' rule example.
4. Conditional probability examples.
5. Homework 3 solutions.
6. Equity gained example.
7. Projects.

OH today will be from 11:10-11:30. None next week!

The exams next week are open book, plus 2 pages of notes.

Bring a calculator and a pen or pencil.

Submit your reviews of the course online via my.ucla.edu.

There's no R stuff on the final.

Suggested problems from ch 4-7 to look at are 4.5, 4.6, 4.7, 4.8, 4.9, 4.13, 4.14, 4.16, 5.1, 5.2, 5.5, 5.6, 6.2, 6.4, 6.9, 6.10, 6.11, 6.12, 7.1, 7.2, 7.3, 7.4, 7.5, 7.8, 7.13, 7.14, 7.15.

1. Review list.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. $P(AB) = P(A)\ P(B|A)$ [= $P(A)P(B)$ if ind.]
7) Odds ratios.
8) Random variables (RVs).
9) Discrete RVs, and probability mass function (pmf).
10) Expected value.
11) Pot odds calculations.
12) Luck, skill, and deal-making.
13) Variance and SD.
14) Bernoulli RV. [0-1. $\mu = p$, $\sigma = \sqrt{(pq)}$.]
15) Binomial RV. [# of successes, out of n tries. $\mu = np$, $\sigma = \sqrt{(npq)}$.]
16) Geometric RV. [# of tries til 1st success. $\mu = 1/p$, $\sigma = (\sqrt{q}) / p$.]
17) Negative binomial RV. [# of tries til rth success. $\mu = r/p$, $\sigma = (\sqrt{rq}) / p$.]
18) Poisson RV [# of successes in some time interval. [$\mu = \lambda$, $\sigma = \sqrt{\lambda}$.]
19) E(X+Y), V(X+Y) (ch. 7.1).
20) Bayes's rule (ch. 3.4).
20) Continuous RVs
21) Probability density function (pdf)
22) Uniform RV
23) Normal RV
24) Exponential RV
25) Moment generating functions
26) Markov and Chebyshev inequalities
25) Law of Large Numbers (LLN)
26) Central Limit Theorem (CLT)
27) Conditional expectation.
28) Confidence intervals for the sample mean.
29) Fundamental theorem of poker
30) Random Walks, Reflection Principle, Ballot Theorem, avoiding zero
31) Chip proportions and induction.

Basically, we've done all of ch. 1-7 except 6.7.

2. Another random walk example.

Suppose that a \$10 winner-take-all tournament has $32 = 2^5$ players. So, you need to double up 5 times to win. Winner gets \$320.

Suppose that on each hand of the tournament, you have probability $p = 0.7$ to double up, and with probability $q = 0.3$ you will be eliminated. What is your expected profit in the tournament?

Your expected *return* = (\$320) x P(win the tournament) + (\$0) x P(you don't win)

= (\$320) x 0.7^5 = \$53.78. But it costs \$10.

So expected *profit* = \$53.78 - \$10 = \$43.78.

3. Bayes' rule example.

Your opponent raises all-in before the flop. Suppose you think she would do that *80%* of the time with AA, KK, or QQ, and she would do that *30%* of the time with AK or AQ, and 1% of the time with anything else.

Given only this, and not even your cards, what's P(she has AK)?

Given nothing, P(AK) = 16/C(52,2) = 16/1326. P(AA) = C(4,2)/C(52,2) = 6/1326.

Using Bayes' rule,

$$P(\text{AK} \mid \text{all-in}) = \frac{P(\text{all-in} \mid \text{AK}) * P(\text{AK})}{P(\text{all-in}|\text{AK})P(\text{AK}) + P(\text{all-in}|\text{AA})P(\text{AA}) + P(\text{all-in}|\text{KK})P(\text{KK}) + \dots},$$

$$= \frac{30\% \times 16/1326}{\underbrace{[30\%\text{x}16/1326]}_{(\text{AK})} + \underbrace{[80\%\text{x}6/1326]}_{(\text{AA})} + \underbrace{[80\%\text{x}6/1326]}_{(\text{KK})} + \underbrace{[80\%\text{x}6/1326]}_{(\text{QQ})} + \underbrace{[30\%\text{x}16/1326]}_{(\text{AQ})} + \underbrace{[1\%\ (1326-16-6-6-6-16)/1326)]}_{(\text{anything else})}}.$$

= **13.06%.** Compare with 16/1326 ~ 1.21%.

4. Conditional prob. examples.

Approximate P(SOMEONE has AA, given you have KK)? Out of your 8 opponents?

Note that given that you have KK,

P(player 2 has AA & player 3 has AA)

= P(player 2 has AA) x P(player 3 has AA | player 2 has AA)

= choose(4,2) / choose(50,2) x 1/choose(48,2)

= 0.0000043, or 1 in 230,000.

So, very little overlap! Given you have KK,

P(someone has AA) = P(player2 has AA or player3 has AA or … or pl.9 has AA)

~ P(player2 has AA) + P(player3 has AA) + … + P(player9 has AA)

= 8 x choose(4,2) / choose(50,2) = 3.9%, or 1 in **26.**

What is **exactly** P(SOMEONE has an Ace | you have KK)? (8 opponents)

(or more than one ace)

Given that you have KK, P(SOMEONE has an Ace) = 100% - P(nobody has an Ace).

And P(nobody has an Ace) = choose(46,16)/choose(50,16)

= **20.1%.**

So P(SOMEONE has an Ace) = **79.9%.**

4. More conditional probability examples.

P(You have AK | you have exactly one ace)?

= P(You have AK and exactly one ace) / P(exactly one ace)

= P(AK) / P(exactly one ace)

= (16/C(52,2)) ÷ (4x48/C(52,2))

= 4/48 = 8.33%.

P(You have AK | you have at least one ace)?

= P(You have AK and at least one ace) / P(at least one ace)

= P(AK) / P(at least one ace)

= (16/C(52,2)) ÷ (((4x48 + C(4,2))/C(52,2)) ~ 8.08%.

P(You have AK | your FIRST card is an ace)?

= 4/51 = 7.84%.

5. Answers to hw3.

6.6. If X is Pareto, with a = 1 and b = 3, then $E(X) = \int y f(y) dy = \int y\ (b/a)\ (a/y)^{b+1}\ dy = \int 3y\text{-}3\ dy = -3/2\ y^{-2}\ ]y=1 \text{ to } \infty$
$= 0 + 3/2\ (1)^{-2} = 1.5$.
$E(X^2) = \int y^2\ (b/a)\ (a/y)^{b+1}\ dy = \int 3y^{-2}\ dy = -3\ y^{-2}\ ]y=1 \text{ to } \infty = 0 + 3 = 3$.
Thus, $V(X) = E(X^2) - (E(X))^2 = 3 - (1.5)^2 = 0.75$.

6.12. X and Y are ind, X is exp with mean 1/2, Y=1 with prob 1/3 and 2 with prob 2/3, and Z = XY. Find a) the pdf of Z, b) the expected value of Z, and c) the sd of Z.

a) Let F be the cdf of Z. $F(c) = P(XY \le c) = P(X \le c/Y) = 1/3\ P(X \le c) + 2/3\ P(X \le c/2)$
$= 1/3\ [1-\exp(-2c)] + 2/3[1-\exp(-c)]$, for $c \ge 0$.
Taking the derivative, $f(c) = F'(c) = 2/3 \exp(-2c) + 2/3 \exp(-c)$, for $c \ge 0$.

b) The expected value of Z is $\int c\ f(c)\ dc = 2/3 \int c \exp(-2c)\ dc + 2/3 \int c \exp(-c)\ dc$.
Integrating by parts,
$E(Z) = \int v du = uv - \int v du$, where u = c and dv = exp(-2c)dc or exp(-c)dc, v = -exp(-2c)/2, or v = -exp(-c),
so $E(Z) = 2/3\ [-c \exp(-2c)/2 + \int \exp(-2c)/2\ dc] + 2/3\ [-c \exp(-c) + \int \exp(-c) dc]$
$= 2/3\ [-c \exp(-2c)/2 - \exp(-2c)/4 - c \exp(-c) - \exp(-c)]$, evaluated from c = 0 to infinity,
and at infinity everything converges to 0, so it's $-2/3\ [0 - 1/4 - 0 - 1] = 2/3 * 5/4 = 5/6$.

Alternatively, without integrating by parts, we know that $\int c\ [2\exp(-2c)]\ dc$ is the expected value of an exponential random variable with parameter lambda = 2, and this expected value is 1/2. Similarly, $\int c \exp(-c)\ dc$ is the expected value of an exponential with lambda = 1, which is 1.
So, $E(Z) = 2/3 \int c \exp(-2c)\ dc + 2/3 \int c \exp(-c)\ dc$
$= 2/3\ (1/2) \int c\ [2\exp(-2c)]\ dc + 2/3\ (1)$
$= 2/3\ (1/2)\ (1/2) + 2/3 = 5/6$.

You could also use E(XY) = E(X)E(Y).

6.12 c) Var(Z) = E(Z^2) - (5/6)^2.

E(Z^2) = ∫c^2 f(c) dc = 2/3 ∫c^2 exp(-2c) dc + 2/3 ∫c^2 exp(-c) dc.

One could do this by integrating by parts, or one could note that if X is exponential (lambda), then E(X^2) = 2/lambda^2, so ∫c^2 [2exp(-2c)] dc = 2/(2^2) = 1/2, and ∫c^2 exp(-c) dc 2/(1^2) = 2.

Thus, E(Z^2) = ∫c^2 f(c) dc = 2/3 ∫c^2 exp(-2c) dc + 2/3 ∫c^2 exp(-c) dc

= 2/3 (1/2) ∫c^2 [2exp(-2c)] dc + 2/3 ∫c^2 exp(-c) dc

= 2/3 (1/2) (1/2) + 2/3 (2)

= 1/6 + 4/3 = 1.5. So, Var(Z) = 1.5 - (5/6)^2 = 54/36 - 25/36 = 29/36. SD(Z) = √(29/36) ~ 0.898.

6.14. Let X be exponential with parameter λ. $ø(t) = E(\exp(tX)) = \int \exp(ty)\, \lambda \exp(-\lambda y)dy$ from 0 to ∞

$= \lambda \int \exp(y(t-\lambda))dy = \lambda \exp(y(t-\lambda))/(t-\lambda) = \lambda/(\lambda-t)$.

Thus $ø'(t) = \lambda/(\lambda-t)^2$ and $E(X) = ø'(0) = \lambda/(\lambda-0)^2 = 1/\lambda$.

$ø''(t) = 2\lambda/(\lambda-t)^3$ so $E(X^2) = ø''(0) = 2\lambda/(\lambda-0)^3 = 2/\lambda^2$.

$V(X) = E(X^2) - [E(X)]^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$.

7.2. X = # of face cards, Y = # of kings. What are a) E(Y)? b) E(Y|X)? c) P{E[Y|X] = 2/3}?

a) E(Y) = 0 * P(0 kings) + 1 * P(1 king) + 2 * P(2 kings)
= 0 + 1 * 4 * 48 / C(52,2) + 2 * C(4,2)/C(52,2) = 204/1326 ~ 15.38%.
b) If X = 0, then Y = 0, so E[Y | X=0] = 0.
E[Y | X = 1] = 0*P(Y=0 | X=1) + 1 * P(Y = 1 | X = 1)
= 0 + 1*P(Y = 1 & X = 1)/P(X=1) = {4 * 40 / C(52,2)} ÷ {12 * 40 / C(52,2)} = 1/3.
E[Y | X = 2] = 0*P(Y=0|X=2) + 1 * P(Y = 1 | X = 2) + 2 P(Y = 2 | X = 2)
= 0 + 1*P(Y = 1 and X = 2)/P(X=2) + 2*P(Y = 2 and X = 2) / P(X=2)
= 4*8/C(52,2) ÷ C(12,2)/C(52,2) + 2*C(4,2)/C(52,2) ÷ C(12,2)/C(52,2)
= 32/66 + 12/66 = 44/66 = 2/3.
c) P{E[Y|X] = 2/3} = P(X = 2) = C(12,2)/C(52,2) ~ 4.98%.

7.8. Negreanu lost $1.7 million in 1250 hands, with an SD per hand of $30,000. a) Find a 95% CI for μ, and b) If Negreanu keeps losing at this rate, how many more hands til the 95% CI doesn't contain 0?

a) A 95% CI for μ = -$1.7 million/1250 +/- 1.96 ($30,000)/√1250 = -$1360 +/- about 1663.
b) 1.96 ($30,000)/√n = 1360, so √n = 1.96(30,000)/1360, and n = {1.96(30,000)/1360}^2 ~ 1869.
He's already played 1250, so he needs about 619 more hands at this rate til the CI doesn't contain 0.

7.14. You have 2 chips and your opponent has 4 chips. p = P(you gain a chip). a) If p = 0.52, what is P(win tournament)? b) Find p so that P(win tournament) = 0.5. c) If p = 0.75 and your opponent has 10 chips left, what is P(win tournament)? What if your opponent has 1,000 chips left?

a) By Theorem 7.6.8, P(win tournament) = (1-r^k)/(1-r^n), where r = q/p = 0.48/0.52 ~ 92.31%. So, P(win tournament) = (1-92.31%^2) / (1-92.31%^6) ~ 38.79%.

b) We want to find p so that (1-r^2)/(1-r^6) = 1/2. That is, 2(1-r^2) = 1 - r^6, so -r^6 + 2r^2 - 1 = 0. Letting x = r^2, this means -x^3 + 2x -1 = 0, so (x-1)(-x^2 - x + 1) = 0. There are 3 solutions to this: x = 1, or
x^2 + x - 1 = 0, so x = (-1 +/- √(1 +4))/2 = √5/2 - 1/2 ~ 0.618, or -√5/2 - 1/2 ~ -1.618.
x = r^2, so r can't be -1.618. Also, r can't be 1 because r = q/p, so r = 1 implies q = p = 0.5, which is not allowed in the conditions of Theorem 7.6.8. So, x must be 0.618 and r = √0.618 = 0.786. (1-p)/p = 0.786, so 1-p = p(0.786), p(1+0.786) = 1. p = 1/(1.786) = 55.99%.

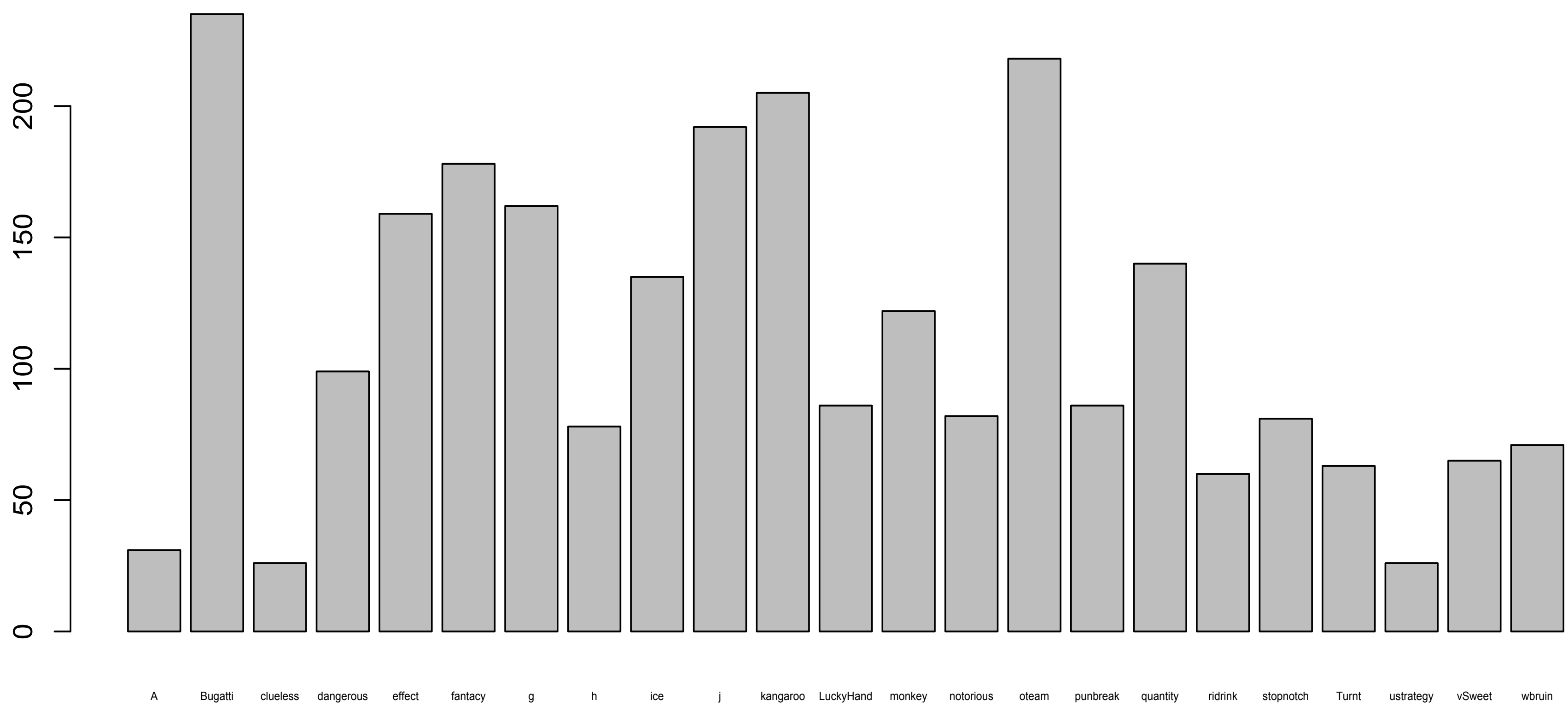
c) If p = 0.75, then r = q/p = 0.25/0.75 = 1/3.
If the opponent has 10 chips, then P(win tournament) = (1-r^2)/(1-r^12) = (1-1/3^2) / (1-1/3^12) ~ 88.88906%.
If the opponent has 1000 chips, then P(win tournament) = (1-1/3^2) / (1-1/3^1002) ~ 88.88889%.

6. Equity gained example.

You have Qc Qd. I have 10s 9s. Board is 10d 8c 7c 4c. Pot is $5.
The river is 2d, you bet $3, and I call.
On the river, how much equity did you gain by luck and how much equity did you gain by skill?

Equity gained by luck on river = your equity when 2d is exposed – your equity on turn
= 100% ($5) - 35/44 ($5) = $1.02.
Why 35/44? I can win with a 10, 9, 6, or J that's not a club. There are 1 + 2 + 3 + 3 = 9 of these cards, so the remaining 35 cards give you the win.

Equity gained by skill on river = increase in pot on river * P(you win) – your cost
= $6 * 100% - $3
= $3.



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