

Stat 100a: Introduction to Probability.

Outline for the day:

1. Collect hw.
2. Gus vs. Daniel.
3. $P(4 \text{ of a kind})$ and related problems.
4. Bayes's rule.
5. Expected value and pot odds.
6. Violette vs. Elezra pot odds example.
7. SD and variance.
8. Markov and Chebyshev inequalities.
9. Luck and skill.
10. Expected value and heads up with AA or 55.

Exam 1 is on everything up to and including chapter 4.

Hw2 is online, due Mon Aug 22, 4.6, 4.8, 4.16, 5.2, 5.6.

Read through chapter 5!



1. HAND IN HW!

2. High Stakes Poker: Daniel vs. Gus

3. P(4 of a kind eventually).

Suppose you're all in next hand, no matter what cards you get.

P(eventually make 4-of-a-kind)? [including case where all 4 are on board]

Trick: just forget card order, and consider all collections of 7 cards.

Out of $\text{choose}(52,7)$ different combinations, each equally likely, how many of them involve 4-of-a-kind?

13 choices for the 4-of-a-kind.

For each such choice, there are $\text{choose}(48,3)$ possibilities for the other 3 cards.

So, $P(4\text{-of-a-kind}) = 13 * \text{choose}(48,3) / \text{choose}(52,7) \sim 0.168\%$, or 1 in **595**.

P(flop 4-of-a-kind) = $13 * 48 / \text{choose}(52,5) = 0.024\% = 1$ in **4165.**

P(flop 4-of-a-kind | pocket pair)?

No matter which pocket pair you have, there are $\text{choose}(50,3)$ possible flops, each equally likely, and how many of them give you 4-of-a-kind?

48. (e.g. if you have $7\spadesuit 7\heartsuit$, then need to flop $7\diamondsuit 7\clubsuit x$, & there are 48 choices for x)

So $P(\text{flop 4-of-a-kind} | \text{pp}) = 48 / \text{choose}(50,3) = 0.245\% = 1$ in **408**.

4. Bayes's rule, p49-52.

Suppose that B_1, B_2, \dots, B_n are disjoint events and that exactly one of them must occur.

Suppose you want $P(B_1 | A)$, but you only know $P(A | B_1), P(A | B_2), \dots$, and you also know $P(B_1), P(B_2), \dots, P(B_n)$.

Bayes' Rule: If B_1, \dots, B_n are disjoint events with $P(B_1 \text{ or } \dots \text{ or } B_n) = 1$, then

$$P(B_i | A) = P(A | B_i) * P(B_i) \div [\sum P(A | B_j)P(B_j)].$$

Why? Recall: $P(X | Y) = P(X \& Y) \div P(Y)$. So $P(X \& Y) = P(X | Y) * P(Y)$.

$$\begin{aligned} P(B_1 | A) &= P(A \& B_1) \div P(A) \\ &= P(A \& B_1) \div [P(A \& B_1) + P(A \& B_2) + \dots + P(A \& B_n)] \\ &= P(A | B_1) * P(B_1) \div [P(A | B_1)P(B_1) + P(A | B_2)P(B_2) + \dots + P(A | B_n)P(B_n)]. \end{aligned}$$

Bayes's rule, continued.

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$$P(B_i | A) = P(A | B_i) * P(B_i) \div [\sum P(A | B_j)P(B_j)].$$

See example 3.4.1, p50. If a test is 95% accurate and 1% of the pop. has a condition, then given a random person from the population,

$P(\text{she has the condition} | \text{she tests positive})$

$$= P(\text{cond} | +)$$

$$= P(+ | \text{cond}) P(\text{cond}) \div [P(+ | \text{cond}) P(\text{cond}) + P(+ | \text{no cond}) P(\text{no cond})]$$

$$= 95\% \times 1\% \div [95\% \times 1\% + 5\% \times 99\%]$$

$$\sim 16.1\%.$$

Tests for rare conditions must be extremely accurate.

Bayes' rule example.

Suppose $P(\text{your opponent has the nuts}) = 1\%$, and $P(\text{opponent has a weak hand}) = 10\%$.

Your opponent makes a huge bet. Suppose she'd only do that with the nuts or a weak hand, and that $P(\text{huge bet} \mid \text{nuts}) = 100\%$, and $P(\text{huge bet} \mid \text{weak hand}) = 30\%$.

What is $P(\text{nuts} \mid \text{huge bet})$?

$P(\text{nuts} \mid \text{huge bet}) =$

$$\frac{P(\text{huge bet} \mid \text{nuts}) * P(\text{nuts})}{$$

$$P(\text{huge bet} \mid \text{nuts}) P(\text{nuts}) + P(\text{huge bet} \mid \text{horrible hand}) P(\text{horrible hand})$$

$$= \frac{100\% * 1\%}{$$

$$100\% * 1\% + 30\% * 10\%$$

$$= \mathbf{25\%}.$$

5. EXPECTED VALUE AND POT ODDS CALCULATIONS.

Suppose someone bets (or raises) you, going all-in. What should your chances of winning be in order for you to correctly call?

Let B = the amount bet to you, i.e. the additional amount you'd need to put in if you want to call. So, if you bet 100 & your opponent with 800 left went all-in, $B = 700$.

Let POT = the amount in the pot right now (including your opponent's bet).

Let p = your probability of winning the hand if you call. So prob. of losing = $1-p$.

Let $CHIPS$ = the number of chips you have right now.

$$\begin{aligned}\text{If you call, then } E[\text{your chips at end}] &= (CHIPS - B)(1-p) + (CHIPS + POT)(p) \\ &= CHIPS(1-p+p) - B(1-p) + POT(p) = CHIPS - B + Bp + POTp\end{aligned}$$

$$\text{If you fold, then } E[\text{your chips at end}] = CHIPS.$$

You want your expected number of chips to be maximized, so it's worth calling if $-B + Bp + POTp > 0$, i.e. if $p > B / (B+POT)$.

5. EXPECTED VALUE AND POT ODDS CALCULATIONS.

Some reasons why Expected Value applies to poker:

- Tournaments: some game theory results suggest that, in symmetric, winner-take-all games, the optimal strategy is the one which uses the *myopic rule*: that is, given any choice of options, always choose the one that maximizes your *expected value*.
- Laws of large numbers: Some statistical theory indicates that, if you repeat an experiment over and over repeatedly, your long-term average will ultimately converge to the *expected value*. So again, it makes sense to try to maximize expected value when playing poker (or making deals).
- Checking results: A great way to check whether you are a long-term winning or losing player, or to verify if a certain strategy works or not, is to check whether the sample mean is positive and to see if it has converged to the *expected value*.

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5. Pot odds and expected value, continued.

From previous slide, to call an all-in, need $P(\text{win}) > B \div (B + \text{pot})$.

Expressed as an *odds ratio*, this is sometimes referred to as *pot odds* or *express odds*.

If the bet is not all-in & another betting round is still to come, need

$$P(\text{win}) > \text{wager} \div (\text{wager} + \text{winnings}),$$

where $\text{winnings} = \text{pot} + \text{amount you'll win on later betting rounds}$,

$\text{wager} = \text{total amount you will wager including the current round \& later rounds}$,
assuming no folding.

The terms *Implied-odds* / *Reverse-implied-odds* describe the cases where
 $\text{winnings} > \text{pot}$ or where $\text{wager} > B$, respectively. See p66.

You will not be tested on implied or reverse implied odds in this course.

Example: 2006 World Series of Poker (WSOP). ♠ ♣ ♥ ♦

Blinds: 200,000/400,000, + 50,000 ante.

Jamie Gold (4♠ 3♣): 60 million chips. Calls.

Paul Wasicka (8♠ 7♠): 18 million chips. Calls.

Michael Binger (A♦ 10♦): 11 million chips. Raises to \$1,500,000.

Gold & Wasicka call. (pot = 4,650,000) Flop: 6♠ 10♣ 5♠.

- Wasicka checks, Binger bets \$3,500,000. (pot = 8,150,000)
- Gold moves all-in for 16,450,000. (pot = 24,600,000)
- Wasicka folds. Q: Based on expected value, should he have called?

If Binger will fold, then Wasicka's chances to beat Gold must be at least
 $16,450,000 / (24,600,000 + 16,450,000) = 40.1\%$.

If Binger calls, it's a bit complicated, but basically Wasicka's chances must be at least
 $16,450,000 / (24,600,000 + 16,450,000 + 5,950,000) = 35.0\%$.

6. Pot odds ex. *Poker Superstars Invitational Tournament*, FSN, October 2005.

Ted Forrest: 1 million chips

Freddy Deeb: 825,000

Blinds: 15,000 / 30,000

Cindy Violette: 650,000

Eli Elezra: 575,000

* Elezra raises to 100,000

* Forrest folds.

* Deeb, the small blind, folds.

* Violette, the big blind with K♦ J♦, calls.

* The flop is: 2♦ 7♣ A♦

* Violette bets 100,000. (pot = 315,000).

* Elezra raises all-in to 475,000. (pot = 790,000).

So, it's 375,000 more to Violette. She folds.

Q: Based on expected value, should she have called?

Her chances must be at least $375,000 / (790,000 + 375,000) = 32\%$.

Violette has K♦ J♦. The flop is: 2♦ 7♣ A♦.

Q: Based on expected value, should she have called?

Her chances must be at least $375,000 / (790,000 + 375,000) = 32\%$.

vs. AQ: 38%. AK: 37% AA: 26% 77: 26% A7: 31%
 A2: 34% 72: 34% TT: 54% T9: 87% 73: 50%

Harrington's principle: always assume at least a 10% chance that opponent is bluffing.

Bayesian approach: average all possibilities, weighting them by their likelihood.

Violette has K♦ J♦. The flop is: 2♦ 7♣ A♦.

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vs. AQ: 38%. AK: 37% AA: 26% 77: 26% A7: 31%
 A2: 34% 72: 34% TT: 54% T9: 87% 73: 50%

Harrington's principle: always assume at least a 10% chance that opponent is bluffing.

Bayesian approach: average all possibilities, weighting them by their likelihood.

Reality: Elezra had 7♦ 3♦. Her chances were 51%. Bad fold.

What was her prob. of winning (given just her cards and Elezra's, and the flop)?

Of $\text{choose}(45,2) = 990$ combinations for the turn & river, how many give her the win?

First, how many outs did she have? eight ♦s + 3 kings + 3 jacks = 14.

She wins with (out, out) or (out, nonout) or (non-♦ Q, non-♦ T)

$$\text{choose}(14,2) + 14 \times 31 + 3 \times 3 = 534$$

but not (k or j, 7 or non-♦ 3) and not (3♦, 7 or non-♦ 3)

$$- 6 \times 4 - 1 \times 4 = 506.$$

So the answer is $506 / 990 = 51.1\%$.

7. Variance and SD.

Expected Value: $E(X) = \mu = \sum k P(X=k)$.

Variance: $V(X) = \sigma^2 = E[(X - \mu)^2]$. Turns out this = $E(X^2) - \mu^2$.

Standard deviation = $\sigma = \sqrt{V(X)}$. Indicates how far an observation would *typically* deviate from μ .

Examples:

Game 1. Say $X = \$4$ if red card, $X = \$-5$ if black.

$$E(X) = (\$4)(0.5) + (\$-5)(0.5) = -\$0.50.$$

$$E(X^2) = (\$4^2)(0.5) + (\$-5^2)(0.5) = (\$16)(0.5) + (\$25)(0.5) = \$20.5.$$

$$\text{So } \sigma^2 = E(X^2) - \mu^2 = \$20.5 - \$-0.50^2 = \$20.25. \quad \sigma = \$4.50.$$

Game 2. Say $X = \$1$ if red card, $X = \$-2$ if black.

$$E(X) = (\$1)(0.5) + (\$-2)(0.5) = -\$0.50.$$

$$E(X^2) = (\$1^2)(0.5) + (\$-2^2)(0.5) = (\$1)(0.5) + (\$4)(0.5) = \$2.50.$$

$$\text{So } \sigma^2 = E(X^2) - \mu^2 = \$2.50 - \$-0.50^2 = \$2.25. \quad \sigma = \$1.50.$$

8. Markov and Chebyshev inequalities. Ch 4.6.

The Markov inequality states

If X takes only non-negative values, and c is any number > 0 , then

$$P(X \geq c) \leq E(X)/c.$$

Proof. The discrete case is given on p82.

Here is a proof for the case where X is continuous with pdf $f(y)$.

$$\begin{aligned} E(X) &= \int y f(y) dy \\ &= \int_0^c y f(y) dy + \int_c^\infty y f(y) dy \\ &\geq \int_c^\infty y f(y) dy \\ &\geq \int_c^\infty c f(y) dy \\ &= c \int_c^\infty f(y) dy \\ &= c P(X \geq c). \end{aligned}$$

Thus, $P(X \geq c) \leq E(X) / c$.

The Chebyshev inequality states

For any random variable Y with expected value μ and variance σ^2 , and any real number $a > 0$,
 $P(|Y - \mu| \geq a) \leq \sigma^2 / a^2$.

Proof. The Chebyshev inequality follows directly from the Markov equality by letting $c = a^2$ and $X = (Y - \mu)^2$.

Examples of the use of the Markov and Chebyshev inequalities are on p83.

9. Luck and skill in poker. pp 71-79. ♠ ♣ ♥ ♦

Let equity = your expected portion of the pot after the hand, assuming no future betting.

= your expected number of chips after the hand - chips you had before the hand

= pot * p, where p = your probability of winning if nobody folds.

I define luck as the equity gained during the dealing of the cards.

Skill = equity gained during the betting rounds.

Example.

You have Q♣ Q♦. I have 10♠ 9♠. Board is 10♦ 8♣ 7♣ 4♣. Pot is \$5.

The river is 2♦, you bet \$3, and I call.

On the river, how much equity did you gain by luck and how much by skill?

Equity gained by luck on river = your equity when 2♦ is exposed – your equity on turn

= 100% (\$5) - 35/44 (\$5) = \$1.02.

Why 35/44? I can win with a 10, 9, 6, or J that is not a club. There are 1 + 2 + 3 + 3 = 9 of these cards, so the remaining 35 cards give you the win.

Equity gained by skill on river = your equity after all the betting is over - your equity when the 2♦ is dealt

= increase in pot on river * P(you win) - your cost

= \$6 * 100% - \$3 = \$3.

9. Luck and skill in poker, continued. ♠ ♣ ♥ ♦

Example.

You have Q♣ Q♦. I have 10♠ 9♠. Board is 10♦ 8♣ 7♣ 4♣. Pot is \$5.

The river is 2♦, you bet \$3, and I call.

On the river, how much equity did you gain by luck and how much by skill?

Alternatively, let x = the number of chips you have after your \$3 bet on the river.

Before this bet, you had $x + \$3$ chips.

Equity gained by skill on river = your equity after all the betting is over - your equity when the 2♦ is dealt

= your expected number of chips after all the betting is over – your expected number of chips when the 2d is dealt

= (100%)($x + \$11$) – (100%)($x + \$3 + \5)

= \$3.

10. Heads up with AA.

Dan Harrington says that, “with a hand like AA, you really want to be one-on-one.” True or false?

* Best possible pre-flop situation is to be all in with AA vs A8, where the 8 is the same suit as one of your aces, in which case you're about 94% to win. (the 8 could equivalently be a 6, 7, or 9.) If you are all in for \$100, then your expected holdings afterwards are \$188.

a) In a more typical situation: you have AA against TT. You're 80% to win, so your expected value is \$160.

b) Suppose that, after the hand vs TT, you get QQ and get up against someone with A9 who has more chips than you do. The chance of you winning this hand is 72%, and the chance of you winning both this hand and the hand above is 58%, so your expected holdings after both hands are \$232:
you have 58% chance of having \$400, and 42% chance to have \$0.

c) Now suppose instead that you have AA and are all in against 3 callers with A8, KJ suited, and 44. Now you're 58.4% to quadruple up. So your expected holdings after the hand are \$234, and the situation is just like (actually slightly better than) #1 and #2 combined: 58.4% chance to hold \$400, and 41.6% chance for \$0.

* So, being all-in with AA against 3 players is much better than being all-in with AA against one player: in fact, it's about like having two of these lucky one-on-one situations.

What about with a low pair?

- a) You have \$100 and 55 and are up against A9. You are 56% to win, so your expected value is \$112.
- b) You have \$100 and 55 and are up against A9, KJ, and QJs. Seems pretty terrible, doesn't it? But you have a probability of 27.3% to quadruple, so your expected value is
- $$0.273 \times \$400 = \$109. \text{ About the same as \#1!}$$

[For these probabilities, see

http://www.cardplayer.com/poker_odds/texas_holdem]