Stat 100a, Introduction to Probability.

<u>Outline for the day:</u>

1. A quick fact about normals.
2. Conditional expectation.
3. Law of Large Numbers.
4. Bivariate and marginal density.

hw3 is due Tue Jul28.

http://www.stat.ucla.edu/~frederic/100A/S20

1. A quick fact about normals.

If X and Y are independent and both are normal, then X+Y is normal, and so are –X and –Y.

2. Conditional expectation, E(Y | X), ch. 7.2.
Suppose X and Y are discrete.
Then E(Y | X=j) is defined as $\sum_k$ k P(Y = k | X = j), just as you'd think.
E(Y | X) is a **random variable** such that E(Y | X) = E(Y | X=j) whenever X = j.

For example, let X = the # of spades in your hand, and Y = the # of clubs in your hand.
a) What's E(Y)? **b)** What's E(Y|X)? **c)** What's P(E(Y|X) = 1/3)?

a. E(Y) = 0P(Y=0) + 1P(Y=1) + 2P(Y=2)
= 0 + 13x39/C(52,2) + 2 C(13,2)/C(52,2) = 0.5.

b. X is either 0, 1, or 2. If X = 0, then E(Y|X) = E(Y | X=0) and
E(Y | X=0) = 0 P(Y=0 | X=0) + 1 P(Y=1 | X=0) + 2 P(Y=2 | X = 0)
= 0 + 13x26/C(39,2) + 2 C(13,2) / C(39,2) = **2/3**.
E(Y | X=1) = 0 P(Y=0 | X=1) + 1 P(Y=1 | X=1) + 2 P(Y=2 | X = 1)
= 0 + 13/39 + 2 (0) = **1/3**.
E(Y | X=2) = 0 P(Y=0 | X=2) + 1 P(Y=1 | X=2) + 2 P(Y=2 | X = 2)
= 0 + 1 (0) + 2(0) = **0**.
So E(Y | X = 0) = 2/3, E(Y | X = 1) = 1/3, and E(Y | X = 2) = 0. That's what E(Y|X) is
c. P(E(Y|X) = 1/3) is just P(X=1) = 13x39/C(52,2) ~ 38.24%.

3. Law of Large Numbers (LLN) and the Fundamental Theorem of Poker.

David Sklansky, *The Theory of Poker,* 1987.

> "Every time you play a hand differently from the way you would have played it if you could see all your opponents' cards, they gain; and every time you play your hand the same way you would have played it if you could see all their cards, they lose. Conversely, every time opponents play their hands differently from the way they would have if they could see all your cards, you gain; and every time they play their hands the same way they would have played if they could see all your cards, you lose."

Meaning?

LLN: If X_1, X_2 , etc. are iid with expected value μ and sd σ, then $\overline{X}_n$ ---> μ.

Any short term good or bad luck will ultimately become *negligible* to the sample mean.

However, this does not mean that good luck and bad luck will ultimately cancel out. See p132.

4. Bivariate and marginal density.

Suppose X and Y are random variables.

If X and Y are discrete, we can define the joint pmf $f(x,y) = P(X = x \text{ and } Y = y)$.

Suppose X and Y are continuous for the rest of this page.

Define the bivariate or joint pdf $f(x,y)$ as a function with the properties that $f(x,y) \geq 0$, and for any a,b,c,d,

$$P(a \leq X \leq b \text{ and } c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

The integral $\int_{-\infty}^{\infty} f(x,y)\, dy = f(x)$, the pdf of X, and this function $f(x)$ is sometimes called the *marginal* density of X. Similarly $\int_{-\infty}^{\infty} f(x,y)\, dx$ is the marginal pdf of Y.

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} x\, [\int_{-\infty}^{\infty} f(x,y)\, dy]\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx.$

Just as $P(A|B) = P(AB)/P(B)$, $f(x|y) = f(x,y)/f(y)$.

X and Y are independent iff. $f(x,y) = f_x(x)f_y(y)$.

Now $E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dy\, dx$. This can be useful to find

$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$.

What is $E(X^2Y+e^Y)$? It $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2y+e^y)\, f(x,y)\, dy\, dx$.

Bivariate and marginal density.

Suppose the joint density of X and Y is $f(x,y) = a \exp(x+y)$, for X and Y in (0,1) x (0,1). What is a? What is the marginal density of Y? What type of distribution does X have conditional on Y? What is E(X|Y)? What is the mean of X when Y = .2? Are X and Y independent?

$\iint a \exp(x+y)\, dxdy = 1 = a\iint \exp(x)\exp(y)dxdy = a\int_0^1 \exp(x)\, dx \int_0^1 \exp(y)\, dy = a(e-1)^2$,
so $a = (e-1)^{-2}$.

The marginal density of Y is $f(y) = \int_0^1 a \exp(x+y)\, dx = a \exp(y) \int_0^1 \exp(x)dx = a \exp(y)(e-1) = \exp(y)/(e-1)$.

Conditional on Y, the density of X is $f(x|y) = f(x,y)/f(y) = a \exp(x+y)(e-1)/\exp(y) = \exp(x)/(e-1)$. So X|Y is like an exponential(1) random variable restricted to (0,1).

$E(X|Y) = \int_0^1 x \exp(x)/(e-1)\, dx = 1/(e-1) [x \exp(x) - \int \exp(x)dx] = 1/(e-1) [x \exp(x) – \exp(x)]_0^1 = 1/(e-1) [e – e – 0 + 1] = 1/(e-1)$.
When Y = .2, $E(X|Y) = 1/(e-1)$.

$f(y) = \exp(y)/(e-1)$ and similarly $f(x) = \exp(x)/(e-1)$,
so $f(x)f(y) = \exp(x+y)/(e-1)^2 = f(x,y)$. Therefore, X and Y are independent.