Stat 100a, Introduction to Probability. Rick Paik Schoenberg

Outline for the day:

1. Conditional probability, independence, and multiplication rule.
2. Independence and dependence examples.
3. Negreanu and Elezra.
4. Odds ratios.
5. P(have AA and flop a full house)?
6. P(A♠ after 1st ace)?
7. Random variables.
8. cdf, pmf, and density.
9. Expected value.
10. P(flop 3 of a kind) and P(eventually make 4 of a kind).

No need to zoom for the exams.

Remember, Wed Aug11 class is recorded, not live.

HW1 is due Wed Aug11.

♠ ♣ ♥ ♦

1. Conditional probability, independence, & multiplication rule.

P(A & B) is often written "P(AB)".

"P(A U B)" means P(A or B [or both]).

Conditional Probability:

P(A given B) [written"P(A|B)"] = P(AB) / P(B).

Independent: A and B are "independent" if P(A|B) = P(A).

Fact (*multiplication rule for independent events*):

If A and B are independent, then P(AB) = P(A) x P(B)

Fact (general *multiplication rule*):

P(AB) = P(A) P(B|A)

P(ABC…) = P(A) x P(B|A) x P(C|A&B) …

2. Independence and dependence examples.

Independence: $P(A \mid B) = P(A)$ [and $P(B|A) = P(B)$].

So, when independent, $P(A\&B) = P(A)P(B|A) = P(A)P(B)$.

Reasonable to assume the following are independent:

a) Outcomes on different rolls of a die.

b) Outcomes on different flips of a coin.

c) Outcomes on different spins of a spinner.

d) Outcomes on different poker hands.

e) Outcomes when sampling from a large population.

Ex: P(you get AA on 1st hand and I get AA on 2nd hand)

= P(you get AA on 1st) x P(I get AA on 2nd)

= 1/221 x 1/221 = 1/48841.

P(you get AA on 1st hand and I get AA on 1st hand)

= P(you get AA) x P(I get AA | you have AA)

= 1/221 x 1/(50 choose 2) = 1/221 x 1/1225 = 1/270725.

3. Negreanu and Elezra example: High Stakes Poker, 1/8/07.

Greenstein folds, Todd Brunson folds, Harman folds. Elezra calls $600, Farha (K♠ J♥) raises to $2600, Sheikhan folds. Negreanu calls, Elezra calls. Pot is $8,800.

Flop: 6♠ 10♠ 8♥ .

Negreanu bets $5000. Elezra raises to $15000. Farha folds.

Negreanu thinks for 2 minutes….. then goes all-in for another $88,000.

Elezra: 8♣ 6♣. (Elezra calls. Pot is $214,800.)

Negreanu: A♦ 10♥ .

At this point, the odds on tv show 73% for Elezra and 25% for Negreanu.

They "run it twice". First: 2♠ 4♥. Second time? A♥ 8♦!

P(Negreanu hits an A or 10 on turn & still loses)?

Given both their hands, and the flop, and the first "run", what is P(Negreanu hits an A or 10 on the turn & loses)?

Since he can't lose if he hits a 10 on the turn, it's:

P(A on turn & Negreanu loses)

= P(A on turn) x P(Negreanu loses | A on the turn)

= 3/43 x 4/42

= 0.66% (1 in 150.5)

Note: this is very different from:

P(A or 10 on turn) x P(Negreanu loses),

which would be about 5/43 x 73% = 8.49% (1 in 12)

4. Odds ratios.

Odds ratio of A = P(A)/P(A^c)

Odds *against* A = Odds ratio of A^c = P(A^c)/P(A).

Ex: (from Phil Gordon's *Little Blue Book*, p189)

Day 3 of the 2001 WSOP, $10,000 No-limit holdem championship.

613 players entered. Now 13 players left, at 2 tables.

Phil Gordon's table has 5 other players. Blinds are 3,000/6,000 + 1,000 antes.

Matusow has 400,000; Helmuth has 600,000; Gordon 620,000.

(the 3 other players have 100,000; 305,000; 193,000).

Matusow raises to 20,000. Next player folds.

Gordon's next, in the *cutoff seat* with K♣ K♠ and re-raises to 100,000.

Next player folds. Helmuth goes all-in. Big blind folds. Matusow folds.

Gordon's decision…. Fold!

Odds against Gordon winning, if he called and Helmuth had AA?

What were the odds against Gordon winning, if he called and Helmuth had AA?

P(exactly one K, and no aces) = 2 x C(44,4) / C(48,5) ~ 15.9%.

P(two Kings on the board) = C(46,3) / C(48,5) ~ 0.9%.

[also some chance of a straight, or a flush…]

Using www.cardplayer.com's poker odds calculator,

P(Gordon wins) is about 18%, so the odds against this are:

$P(A^c)/P(A)$ = 82% / 18% = 4.6 (or "*4.6 to 1" or "4.6:1")*.

5. P(you get dealt AA and flop a full house)

= P(you get dealt AA) * P(you flop a full house | AA)

= C(4,2) / C(52,2) * P(triplet or Axx | AA)

= 6/1326 * (12 * C(4,3) + 2*12*C(4,2))/C(50,3)

= .00433%.

P(you are dealt A♦ K♦ and flop a royal flush)? This relates to the unbreakable nuts hw question in a way.

= P(you get dealt A♦ K♦) * P(you flop a royal flush | you have A♦ K♦)

= P(you get dealt A♦ K♦) * P(flop contains Q♦ J♦ 10♦ | you have A♦ K♦)

= 1 / C(52,2) * 1/C(50,3)

= 1 / 25,989,600.

6. Deal til first ace appears. Let X = the *next* card after the ace.

P(X = A♠)? P(X = 2♣)?

(a) How many permutations of the 52 cards are there?

52!

(b) How many of these perms. have A♠ right after the 1st ace?

(i) How many perms of the *other* 51 cards are there?

51!

(ii) For *each* of these, imagine putting the A♠ right after the 1st ace.

1:1 correspondence between permutations of the other 51 cards & permutations of 52 cards such that A♠ is right after 1st ace.

So, the answer to question (b) is 51!.

Answer to the overall question is 51! / 52! = 1/52.

Obviously, same goes for 2♣.

7. Random variables.

A *variable* is something that can take different numeric values.

A *random variable* (X) can take different numeric values with different probabilities.

X is *discrete* if all its possible values can be listed. If X can take any value in an interval like say [0,1], then X is *continuous*.

Ex. Two cards are dealt to you. Let X be 1 if you get a pair, and X is 0 otherwise.

P(X is 1) = 3/51 ~ 5.9%. P(X is 0) ~ 94.1%.

Ex. A coin is flipped, and X=20 if heads, X=10 if tails.

The *distribution* of X means all the information about all the possible values X can take, along with their probabilities.

8. cdf, pmf, and density (pdf).

Any random variable has a *cumulative distribution function* (cdf):

$F(b) = P(X \leq b)$.

If X is discrete, then it has a *probability mass function* (pmf):

$f(b) = P(X = b)$.

Continuous random variables are often characterized by their *probability density functions* (pdf, or *density*):

a function f(x) such that $P(X \text{ is in } B) = \int_B f(x)\, dx$.

9. Expected value.

For a discrete random variable X with pmf f(b), the *expected value* of X = $\sum$ b f(b).

The sum is over all possible values of b.   (continuous random variables later…)

The expected value is also called the *mean* and denoted E(X) or μ.

Ex: 2 cards are dealt to you. X = 1 if pair, 0 otherwise.

P(X is 1) ~ 5.9%, P(X is 0) ~ 94.1%.

E(X) = (1 x 5.9%) + (0 x 94.1%) = 5.9%, or 0.059.

Ex. Coin, X=20 if heads, X=10 if tails.

E(X) = (20x50%) + (10x50%) = 15.

Ex. Lotto ticket. f($10million) = 1/choose(52,6) = 1/20million, f($0) = 1-1/20mil.

E(X) = ($10mil x 1/20million) = $0.50.

The expected value of X represents a *best guess* of X.

Compare with the *sample mean*, $\overline{X} = (X_1 + X_1 + \ldots + X_n) / n$.

Some reasons why Expected Value applies to poker:

- Tournaments: some game theory results suggest that, in symmetric, winner-take-all games, the optimal strategy is the one which uses the *myopic rule*: that is, given any choice of options, always choose the one that maximizes your *expected value*.
- Laws of large numbers: Some statistical theory indicates that, if you repeat an experiment over and over repeatedly, your long-term average will ultimately converge to the *expected value*. So again, it makes sense to try to maximize expected value when playing poker (or making deals).
- Checking results: A great way to check whether you are a long-term winning or losing player, or to verify if a certain strategy works or not, is to check whether the sample mean is positive and to see if it has converged to the *expected value*.

Heads up with AA.

Dan Harrington says that, “with a hand like AA, you really want to be one-on-one.” True or false?

* Best possible pre-flop situation is to be all in with AA vs A8, where the 8 is the same suit as one of your aces, in which case you're about 94% to win. (the 8 could equivalently be a 6,7, or 9.) If you are all in for $100, then your expected holdings afterwards are $188.

a) In a more typical situation: you have AA against TT. You're 80% to win, so your expected value is $160.

b) Suppose that, after the hand vs TT, you get QQ and get up against someone with A9 who has more chips than you do. The chance of you winning this hand is 72%, and the chance of you winning both this hand and the hand above is 58%, so your expected holdings after both hands are $232:
you have 58% chance of having $400, and 42% chance to have $0.

c) Now suppose instead that you have AA and are all in against 3 callers with A8, KJ suited, and 44. Now you're 58.4% to quadruple up. So your expected holdings after the hand are $234, and the situation is just like (actually slightly better than) #1 and #2 combined: 58.4% chance to hold $400, and 41.6% chance for $0.

* So, being all-in with AA against 3 players is much better than being all-in with AA against one player: in fact, it's about like having two of these lucky one-on-one situations.

10. Which is more likely, given no info about your cards:

* flopping 3 of a kind,

or

* eventually making 4 of a kind?

P(flop 3-of-a-kind)?

[including case where all 3 are on board, and *not including full houses*]

<u>Key idea</u>: forget order! Consider all combinations of your 2 cards and the flop.

Sets of 5 cards. Any such combo is equally likely! choose(52,5) different ones.

P(flop 3 of a kind) = # of different 3 of a kinds / choose(52,5)

How many different 3 of a kind combinations are possible?

13 * choose(4,3) different choices for the triple.

For each such choice, there are choose(12,2) choices left for the numbers on the other 2 cards, and for each of these numbers, there are 4 possibilities for its suit.

So, P(flop 3 of a kind) = 13 * choose(4,3) * choose(12,2) * 4 * 4 / choose(52,5)

~ 2.11%, or 1 in 47.3.

P(flop 3 of a kind or a full house) = 13 * choose(4,3) * choose(48,2) / choose(52,5)

~ 2.26%, or 1 in 44.3.

P(eventually make 4-of-a-kind)? [including case where all 4 are on board]

Again, just forget card order, and consider all collections of 7 cards.

Out of choose(52,7) different combinations, each equally likely, how many of them involve 4-of-a-kind?

13 choices for the 4-of-a-kind.

For each such choice, there are choose(48,3) possibilities for the other 3 cards.

So, P(4-of-a-kind) = 13 * choose(48,3) / choose(52,7) ~ 0.168%, or 1 in 595.