Stat 100a, Introduction to Probability. Rick Paik Schoenberg

Outline for the day:

1. Exam notes.
2. Variance and SD.
3. More combinations questions.
4. HW2.
5. Markov and Chebyshev inequalities.
6. Luck and skill in poker.
7. Facts about expected value.
8. Expected number trick.
9. Conditional probability.
10. Review list.
11. Example problems.

I will divide you into teams next week for the R project.
Remember, this Wed Aug11 class is recorded, not live.
HW1 is due Wed Aug11, 950am. Exam 1 will be Mon Aug16, 10am-11:15am.

1. Exam notes.

a. The exam will only be 75 minutes, from 10am to 11:15am. After this exam, class if over. However, after exam 2 on Aug30 there will be lecture.

b. The exam will have 14 multiple choice questions on it. None of the above will always be an option. Final answers will be rounded to 3 significant digits.

c. NO cheating! Answers must be done independently. No communication or internet surfing during the exam.

d. At 10am, I will post the exam on the course website, http://www.stat.ucla.edu/~frederic/100A/sum21 . It will be called exam1.pdf .

e. At 11:15am, you must email me your answers. Write your name and your student ID number at the top of your email. You do not have to show work. Just email me something like

Jane Smith.

ID 102817104

AEB EDD DEB CAA CB.

2. Variance and SD.

Expected Value: $E(X) = \mu = \sum k\ P(X=k)$.

Variance: $V(X) = \sigma^2 = E[(X-\mu)^2]$. Turns out this $= E(X^2) - \mu^2$.

Standard deviation $= \sigma = \sqrt{V(X)}$. Indicates how far an observation would *typically* deviate from μ.

Examples:

Game 1. Say X = \$4 if red card, X = \$-5 if black.

E(X) = (\$4)(0.5) + (\$-5)(0.5) = -\$0.50.

$E(X^2)$ = (\$4²)(0.5) + (\$-5²)(0.5) = (\$16)(0.5) + (\$25)(0.5) = \$20.5.

So $\sigma^2 = E(X^2) - \mu^2$ = \$20.5 - \$-0.50² = \$20.25. **σ = \$4.50**.

Game 2. Say X = \$1 if red card, X = \$-2 if black.

E(X) = (\$1)(0.5) + (\$-2)(0.5) = -\$0.50.

$E(X^2)$ = (\$1²)(0.5) + (\$-2²)(0.5) = (\$1)(0.5) + (\$4)(0.5) = \$2.50.

So $\sigma^2 = E(X^2) - \mu^2$ = \$2.50 - \$-0.50² = \$2.25. **σ = \$1.50**.

3. More combinations questions.

P(flop 4 of a kind)?

Suppose you're all in next hand, no matter what cards you get.

P(flop 4 of a kind) = 13*48 / choose(52,5) = 0.024% = 1 in **4165**.

P(flop 4 of a kind | pocket pair)?

No matter which pocket pair you have, there are choose(50,3) possible flops, each equally likely, and how many of them give you 4-of-a-kind?

48. (e.g. if you have 7♠ 7♥, then need to flop 7♦ 7♣ x, & there are 48 choices for x)

So P(flop 4-of-a-kind | pp) = 48/choose(50,3) = 0.245% = 1 in **408**.

4. HW2.

Homework 2 is problems 4.6, 4.8, 4.26 and 5.2. 4.26 is only in the 2nd edition, but it is also in hw2.pdf. 5.2 is tricky.

On problem 5.2, let Z = the time until you have been dealt a pocket pair and you have also been dealt two black cards.

Consider $P(Z > k)$, and $P(Z > k-1)$. These are actually easier to derive in this case than $P(Z = k)$. Can you get $P(Z = k)$ in terms of these?

5. Markov and Chebyshev inequalities. Ch 4.6.

The Markov inequality states

If X takes only non-negative values, and c is any number > 0, then

$P(X \geq c) \leq E(X)/c$.

Proof. The discrete case is given on page 123.

If X is discrete and nonnegative, then

$$
\begin{aligned}
E(X) &= \sum_b b\, P(X = b) \\
&= \sum_{b<c} b\, P(X = b) + \sum_{b\geq c} b\, P(X = b) \\
&\geq \sum_{b\geq c} b\, P(X = b) \\
&\geq \sum_{b\geq c} c\, P(X = b) \\
&= c \sum_{b\geq c} P(X = b) \\
&= c\, P(X \geq c).
\end{aligned}
$$

Here is a proof for the case where X is continuous with pdf f(y).

$E(X) = \int y\, f(y)\, dy$

$= \int_0^c yf(y)dy + \int_c^\infty yf(y)dy$

$\geq \int_c^\infty yf(y)dy$

$\geq \int_c^\infty cf(y)dy$

$= c \int_c^\infty f(y)dy$

$= c\, P(X \geq c)$.

Thus, $P(X \geq c) \leq E(X) / c$.

Markov and Chebyshev inequalities. Ch 4.6.

The Markov inequality states

If X takes only non-negative values, and c is any number > 0, then

$P(X \geq c) \leq E(X)/c$.

The Chebyshev inequality states

For any random variable Y with expected value μ and variance σ^2, and any real number $a > 0$,

$P(|Y - \mu| \geq a) \leq \sigma^2 / a^2$.

Proof. The Chebyshev inequality follows directly from the Markov equality by letting $c = a^2$ and $X = (Y-\mu)^2$.

Examples of the use of the Markov and Chebyshev inequalities are on p83.

6. Luck and skill in poker. ♠ ♣ ♥ ♦

Let equity = your expected portion of the pot after the hand, assuming no future betting.
= your expected number of chips after the hand - chips you had before the hand
= pot * p, where p = your probability of winning if nobody folds.
I define luck as the expected profit gained during the dealing of the cards,
= equity gained during the dealing of the cards.
Skill = expected profit gained during the betting rounds.

Example.
You have Q♣ Q♦. I have 10♠ 9♠. Board is 10♦ 8♣ 7♣ 4♣. Pot is $5.
The river is 2♦, you bet $3, and I call.
On the river, how much expected profit did you gain by luck and how much by skill?

Expected profit by luck on river = your equity after 2♦ is exposed – your equity just pre-2♦
= 100% ($5) - 35/44 ($5) = $1.02.
Why 35/44? I can win with a 10, 9, 6, or J that is not a club. There are 1 + 2 + 3 + 3 = 9 of these cards, so the remaining 35 cards give you the win.

Expected profit by skill on river
= increase in pot on river * P(you win) - your cost
= $6 * 100% - $3 = $3.

Luck and skill in poker, continued. ♠ ♣ ♥ ♦

Example.

You have Q♣ Q♦. I have 10♠ 9♠. Board is 10♦ 8♣ 7♣ 4♣. Pot is $5.

The river is 2♦, you bet $3, and I call.

On the river, how much expected profit did you gain by luck and how much by skill?

Alternatively, let x = the number of chips you have after your $3 bet on the river.

Before this bet, you had x + $3 chips.

Expected profit gained by skill on river = your equity after all the betting is over - your equity when the 2♦ is dealt

= your expected number of chips after all the betting is over – your expected number of chips when the 2♦ is dealt

= (100%)(x + $11) – (100%)(x + $3 + $5)

= $3.

Lederer and Minieri.

I define luck as the expected profit gained during the dealing of the cards.
Skill = expected profit gained during the betting rounds.

Are there any problems with these definitions?

Bluffing. Ivey and Booth.

Mike Cloud raised to 15,000 with A♣ A♠, Hellmuth called with A♥ K♠, Daniel Negreanu called from the big blind with 6♦ 4♥, and the flop came K♣ 8♥ K♥. Before the flop, the pot was 57,000 chips.

After the flop, all three players checked, the turn was the J♥, Negreanu checked, Cloud bet 15,000, Hellmuth called, and Negreanu folded.

The river was the 7♠, Cloud checked, Hellmuth bet 37,000, and Cloud called. How much expected profit did Hellmuth gain due to luck and how much due to skill on the river?

Answer—When the turn was dealt, Hellmuth's probability of winning in a showdown was 41/42 ~ 97.62%. After the betting on the turn was over, the pot was 87,000 chips. When the 7♠ was revealed on the river, Hellmuth's equity increased from 97.62% × 87,000 = 84,929.4 to 100% × 87,000, for an increase of 2070.6 chips due to luck.

Hellmuth's expected profit gained due to skill on the river is simply 37,000 chips: the pot size increased by 74,000 while Hellmuth had a 100% chance of winning, but the cost to Hellmuth was 37,000, so his profit was 37,000.

7. Facts about expected value.

For any random variable X and any constants a and b,

$E(aX + b) = aE(X) + b$.

Also, $E(X+Y) = E(X) + E(Y)$,

unless $E(X) = \infty$ and $E(Y) = -\infty$, in which case $E(X)+E(Y)$ is undefined.

Thus $\sigma^2 = E[(X-\mu)^2]$

$= E[(X^2 - 2\mu X + \mu^2)]$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$.

8. Expected number trick.

The board consists of 5 cards. Find the expected number of clubs on the board.

Let X_1 = 1 if the 1st card is a club, and 0 otherwise.

Let X_2 = 1 if the 2nd card is a club, and 0 otherwise.

etc.

$X = X_1 + X_2 + X_3 + X_4 + X_5$.

So $E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5)$

$= [¼ (1) + ¾ (0)] \times 5 = 1.25$.

Even though X_1, X_2, X_3, X_4, and X_5 are not independent, nevertheless

$E(X_1 + X_2 + X_3 + X_4 + X_5) = E(X_1)+E(X_2)+E(X_3)+E(X_4)+E(X_5)$.

9. Conditional probability.

When A and B are different outcomes on different collections of cards or different hands, then P(B|A) can often be found directly.

But when A and B are outcomes on the same event, or same card, then sometimes it is helpful to use the definition P(B|A) = P(AB)/P(A).

For example, let A = the event your hole cards are black, and let B = the event your hole cards are clubs.

P(B|A) = P(AB)/P(A) = C(13,2)/C(52,2) / [C(26,2)/C(52,2)].

However, if A is the event your hole cards are black and B is the event the flop cards are all black, then P(B|A) = C(24,3)/C(50,3) directly.

10. Review.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. $P(AB) = P(A)\,P(B|A)$ [$= P(A)P(B)$ if ind.]
7) Odds ratios.
8) Discrete RVs and probability mass function (pmf).
9) Expected value.
10) Pot odds calculations.
11) Luck and skill.
12) Variance and SD.
13) $E(aX+b)$, $E(X+Y)$, $V(X+Y)$.
14) Bayes's rule.
15) Markov and Chebyshev inequalities.

We have basically done all of chapters 1-4.

Example problems.

___ 1. What is the probability that you will be dealt a king and another card of the same suit as the king?

a. 1.69%. b. 3.62%. c. 4.89%. d. 5.02%. e. None of the above.

4 * 12 / C(52,2) = 3.62%.

P(flop an ace high flush)? [where the ace might be on the board]

-- 4 suits

-- one of the cards must be an ace. C(12,4) possibilities for the others.

So P(flop ace high flush) = 4 * C(12,4) / C(52,5)

= 0.0762%, or 1 in **1313**.

P(flop a straight | 87 of different suits in your hand)?

It could be 456, 569, 6910, or 910J. Each has 4*4*4 = 64 suit combinations.

So P(flop a straight | 87) = 64 * 4 / C(50,3)

= 1.31%.

P(flop a straight | 86 of different suits in your hand)?

Now it could be 457, 579, or 7910.

P(flop a straight | 86) = 64 * 3 / C(50,3)

= 0.980%.

Suppose X = 0 with probability ½, 1 with probability ¼, 2 with probability 1/8, and 3 with probability 1/8.
What is E(X)? What is $E(X^2)$? What is Var(X)? What is SD(X)?

E(X) = 0(1/2) + 1(1/4) + 2(1/8) + 3(1/8) = 0.875.

$E(X^2) = 0(1/2) + 1(1/4) + 4(1/8) + 9(1/8) = 1.875$.

$Var(X) = E(X^2) - \mu^2 = 1.875 - 0.875^2 = 1.11$.

$SD(X) = \sqrt{1.11} = 1.053$.

P(flop two pairs).

If you're sure to be all-in next hand, what is P(you will flop two pairs)?

This is a tricky one. Don't double-count (4♠ 4♦ 9♠ 9♦ Q♦) and (9♠ 9♦ 4♠ 4♦ Q♦).

There are choose(13,2) possibilities for the NUMBERS of the two pairs.

For each such choice (such as 4 & 9),

there are choose(4,2) choices for the suits of the lower pair,

and the same for the suits of the higher pair.

So, choose(13,2) * choose(4,2) * choose(4,2) different possibilities for the two pairs.

For each such choice, there are 44 [52 - 8 = 44] different possibilities for your fifth card, so that it's not a full house but simply two pairs. So,

P(flop two pairs) = C(13,2) * C(4,2) * C(4,2) * 44 / C(52,5)

~ 4.75%, or 1 in **21**.

P(flop two pairs).

Here is another way to do it. Find the mistake.

P(flop 2 pairs) = P(pocket pair and flop 2 pairs) + P(no pocket pair and flop 2 pairs)

= P(pocket pair) P(flop 2 pairs | pocket pair) + P(no pocket pair) P(flop 2 pairs | no pocket pair)

= P(pocket pair aa) * P(bbc | aa) + P(ab)*P(abc | ab)

= 13*C(4,2)/C(52,2) * 12*C(4,2)*44/C(50,3) + C(13,2)*4*4/C(52,2) * 3*3*44/C(50,3)

= 2.85%.

What is the problem here?

P(flop two pairs).

Here is another way to do it. Find the mistake.

P(flop 2 pairs) = P(pocket pair and flop 2 pairs) + P(no pocket pair and flop 2 pairs)

= P(pocket pair) P(flop 2 pairs | pocket pair) + P(no pocket pair) P(flop 2 pairs | no pocket pair)

= P(pocket pair aa) * P(bbc | aa) + P(ab)*P(abc | ab)

= 13*C(4,2)/C(52,2) * 12*C(4,2)*44/C(50,3) + C(13,2)*4*4/C(52,2) * **3*3*44**/C(50,3)

= 2.85%.

What is the problem here?

P(flop 2 pairs | no pocket pair) ≠ P(ab)*P(abc | ab). If you have ab, it could come acc or bcc on the flop.

13*C(4,2)/C(52,2) * 12*C(4,2)*44/C(50,3) + C(13,2)*4*4/C(52,2) * **(3*3*44 + 6*11*C(4,2))** /C(50,3)

= 4.75%.